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Teaching and Learning Slide

TDM1113
Discrete Mathematics

Basharoh Abdul Karim
Nur Hazwani Mohd Zanuddin
Norul Farihah Kholiah

TDM1113: DISCRETE MATHEMATICS
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First published 2024

e ISBN 978-629-97432-6-2

Basharoh Abdul Karim
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Norul Farihah Kholiah

Editor: Nur Hazwani Mohd Zanuddin

Published by:
Kolej Poly-Tech MARA Alor Setar,
Tingkat 1, Kompleks Perniagaan MARA Mergong,
Seberang Jalan Putra,
05150 Alor Setar, Kedah

Tel: 04-7309255/57
Email: publication_as@gapps.kptm.edu.my

PREFACE

Assalamualaikum and peace be upon you.

Alhamdulillah, the TDM1113 Discrete Mathematics Teaching and Learning Slide has been successfully published. This slide has been prepared by lecturers from the Department of Mathematics and Computer Science, KPTM Alor Setar. It is based on the latest Discrete Mathematics course syllabus to ensure relevance and alignment with current teaching needs.

This slide has been developed as a valuable reference for both lecturers and students, offering a clear and structured approach to the key concepts of Discrete Mathematics.

I would like to extend my heartfelt thanks to all the lecturers who contributed their expertise, feedback, and support in producing this teaching and learning slide. Your comments and contributions were essential to the success of this project.

Thank you.

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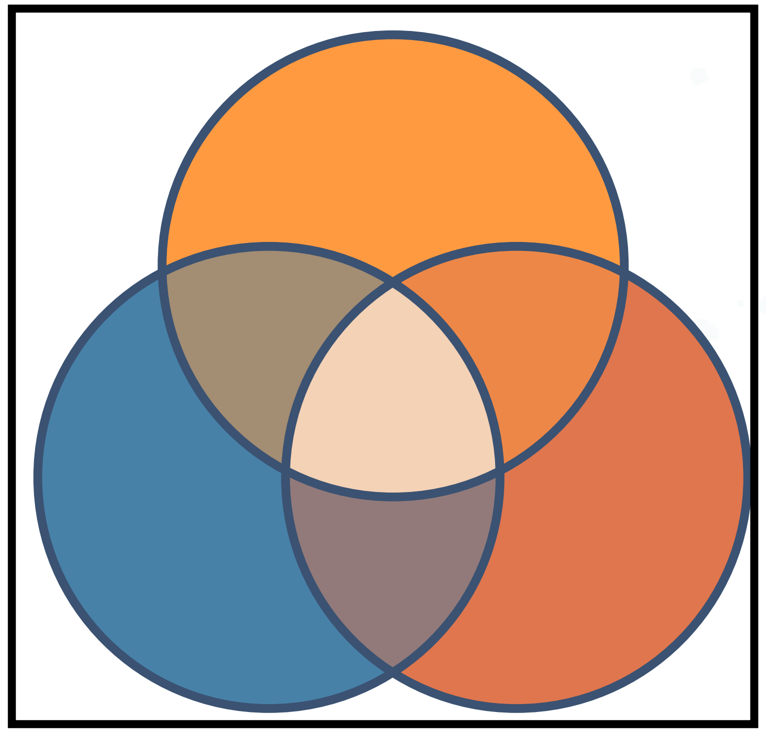
SETS



1.1 Sets of Numbers
1.2 Introduction to Sets
1.3 Types of Sets
1.4 Operations on Sets
1.5 Venn Diagrams

Learning Outcome

At the end of this topic, students should be able to:

1. Understand the basic concept of set theory.
2. Perform operations on sets, including union, intersection, difference, and complement.
3. Use Venn diagrams to visualize and solve problems related to sets and their relationships.

1.1 Sets of Numbers

R Set of all **Real Number**, {x|-∞<x<+∞}

Z Set of all **Integers Numbers** = {...,-3,-2,-1,0,1,2,3,4,....}

W Set of all **Whole Numbers** ={0,1,2,3,4,....}

Q **Rational Numbers -**
Any number in the form of a/b where a and b are integers and b is not equal to 0.
Example : -1/4, 4/2, 1/3, 7/9, 17/99, 5/11

N Set of all **Natural Numbers** = {1,2,3,4,....}

R\Q **Irrational Numbers** -
Real number that cannot be expressed as a ratio of integers.
Example : 22/7, 11/17, √7

Exercise 1

1.Classify the statements below as either True (T) or False (F).

a. -14.92 is an integer.
b. 3/17 is a rational number.
c. √25 is an whole number.
d. 13/4 is a rational number.
e. 15.6666666... is an irrational number.
f. 0 is natural numbers.

2. Classify each number as natural, whole, integer, rational or irrational. Write as many as apply.

a. -22
b. √42

1.2 Introduction to Sets

- A set is a collection of objects called elements or members of a set.
- A set is represented by listing its elements, enclosed in curly brackets { }.
- Sets are typically denoted by capital letters such as A,B,C.
- Elements of sets are written in lowercase or appropriate symbols.

Example 1:

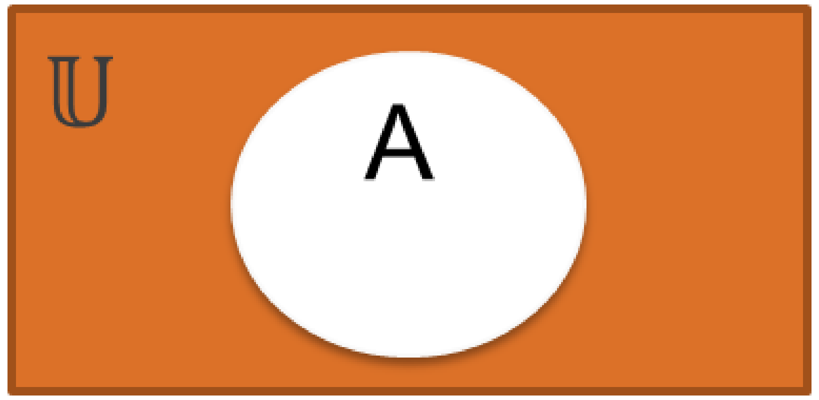
1. A = {1,2,3,4} is a set containing the elements 1, 2, 3, and 4.
2. Set of vowels in the English alphabet, V = {a,e,i,o,u}
3. Set of odd numbers less than 10. E = {1,3,5,7,9}

1.3 Types of Sets

1.3.1 Universal Set, U

- The universal set U is defined as the set consisting of all elements under consideration.
- In Venn diagrams, the universal set is represented as a rectangle enclosing all the subsets.

Example 2 – U is a universal set where A is a subset of U



1.3.2 Members/element of Sets, ∈

- The notation a∈A is used to indicate that the object **a** is a member/element of set A

- If an element x is a member of any set A, it is denoted by x∈A
- if an element y is not a member of set A, it is denoted by y∉A

- **Example 3** – If S = {1,3,4,7}, 1∈S but 5∉S

1.3.3 Empty Set, ∅ or { }

- Empty set / null set The set that contains no elements.
- Notation for empty set is ∅ or { }.

1.3.4 Subsets, ⊆

- Let A and B are two sets. A is a subset of B or that A is contained in B, written as A⊆B if and only if every element of A is also an element of B.
- If set A is not a subset of B, then A⊈B.

Example 4

Let, Set A = {a,b,c,d}

Set B = {b,c,d} and

Set C = {b,c,d,a}

- Here set B is a subset of set A as all the elements of set B is in set A. Hence, we can writeB⊆A.
- Here set C is a subset of a set A as all the elements of set C is equal to set A. Hence, we can writeA⊆B.

1.3.5 Proper Subset, ⊂

- Let A and B are two sets. Set A is proper subset of set B, when set A is a subset of a set B and is not equal to B, and write as $A \subset B$.

Example 5

Let, Set A = {3,5,9}

Set B = {2,3,5,7,9} and

Set C = {5,7,2,3,9}

- Set A is a proper subset of set B. Hence we can write $A \subset B$.
- Set C is a subset of set B but not proper subset of set B.
- Hence we can write $C \subseteq B$ but $C \not\subset B$.

Example 6

Use $\subseteq$, $\nsubseteq$, $\subset$ and $\not\subset$ where appropriate.

Let, A = {a,e,i,o,u}, B = {a,o,u} and C = {a,u,o,i,e},
Hence:

- $B \subseteq A$
- $B \subset A$
- $C \subseteq A$
- $C \not\subset A$
- $A \subseteq C$
- $A \not\subset C$

Example 7

Illustrate the use of ⊆, ⊈, ⊂ or ⊄ . Let A = {1,2,3}, B = {1,2,3,4,5} and C = {1,2,3}.

The relations between pairs of these sets are:

a. A C
b. A C
c. A B
d. A B
e. B A
f. C B
g. C B

1.3.6 Cardinality of Set, |S| or n(S)

- The cardinality of S is denoted by |S| or n(S).
- Cardinality refers to the quantity of members belonging to the set.
- If a set has an infinite number of elements, its cardinality is ∞.

Example 8

a. Let, Set A = {3,5,9}

$$|A| = |\{3,5,9\}| = 3$$

b. Let, Set B = {1,2,3,4,…}

$$|B| = |\{1,2,3,4,\ldots\}| = \infty$$

1.3.7 Power of Sets, P(A)

- The Power set of a set S is the set of all subsets of S including the empty set.
- The formula to determine the number of elements in Power of Sets/ subsets of a set with n elements is $|P(A)| = 2^n$.

Example 9

Given set A = {1,2,3}, thus the number of elements in Power of Sets are $2^n=2^3=8$.

Subsets with zero element - ∅
Subsets with 1 element - {1}, {2}, {3}
Subsets with 2 element - {1,2}, {1,3}, {2,3}
Subsets with 3 element - {1,2,3}
Hence, P(A) = {∅,{1},{2},{3},{1,2},{1,3}, {2,3},{1,2,3}}

Exercise 2

1.Given that set B = {2,5}

a. Determine the total number of subsets in the power set B .
b. Find the power of set B.

2. Given that set C = {a,b,4}

a. Determine the total number of subsets in the power set C .
b. Find the power of set C.

3. Given that set D = {7,9,10}

a. Determine the $|P(D)|$
b. Find the power of set D.

1.4 Operations on Sets

1.4.1 Union of Sets

1.4.2 Intersection of Sets

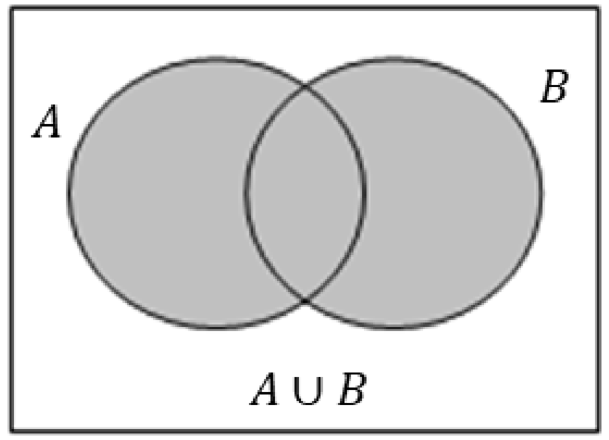
1.4.3 Complement of sets

1.4.4 Disjoint sets

1.4.5 Difference / Subtraction Set

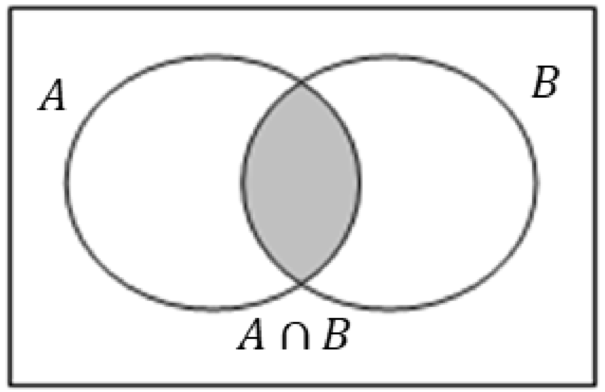
1.4.1 Union of Sets, ∪

- The union of sets A and B (denoted by A∪B) is the set of elements that are present in set A, in set B or both sets.
- Hence, A∪B = {x | x∈A or x∈B}



1.4.2 Intersection of Sets, ∩

- The intersection of sets A and B (denoted by A∩B) is the set of same elements in both A and B.
- Hence, A∩B={x| x∈A and x∈B}.



Example 10

1. If A = {1,2,3,4} and B = {3,4,5,7,9}

   Then,

   $(A \cup B) = \{1,2,3,4,5,7,9\}$

   $(A \cap B) = \{3,4\}$

2. If M = {10,11,15,21,22} and N = {2,4,6,7,9}

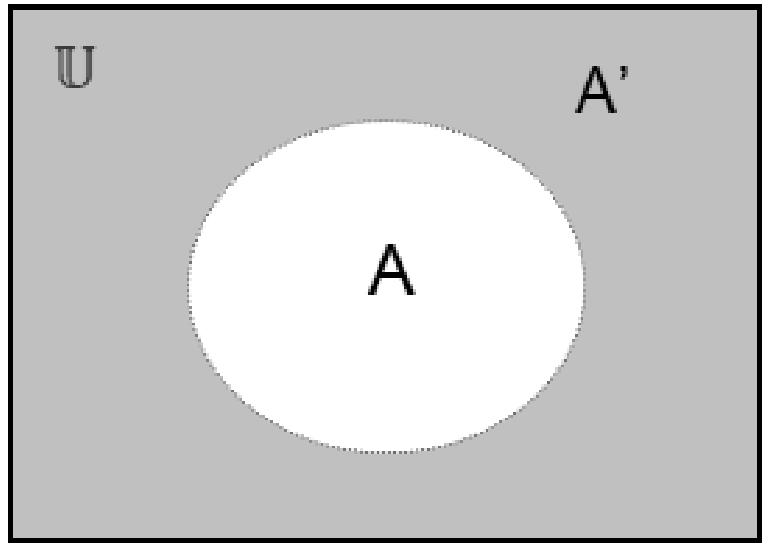
   Then,

   $(M \cup N) = \{2,4,6,7,9,10,11,15,21,22\}$

   $(M \cap N) = \{\ \}$

1.4.3 Complement of Sets.

- If U is a universal set and set A be any subset of U then the complement of A written as $\bar{A}$ or A′ is the set of consisting of all elements in the universal set U which are not in A.
- Hence, A′ = {x | x∈U or x∉A}

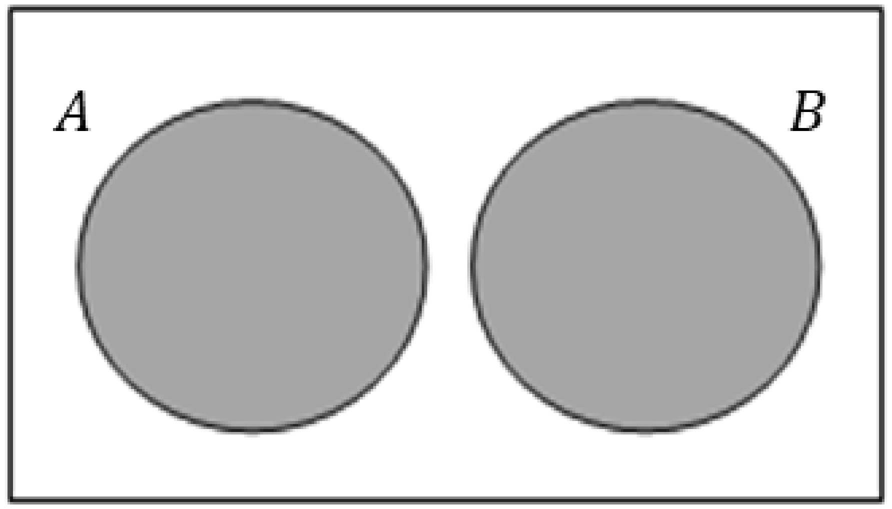


Example 11

If U = {1,2,3,4,5,7,9,10} and A = {5,7,9}
Then, A′ = {1,2,3,4,10}

1.4.4 Disjoint Sets

- If two sets A and B have no elements in common, if A∩B = ∅, then A and B are called disjoint sets.

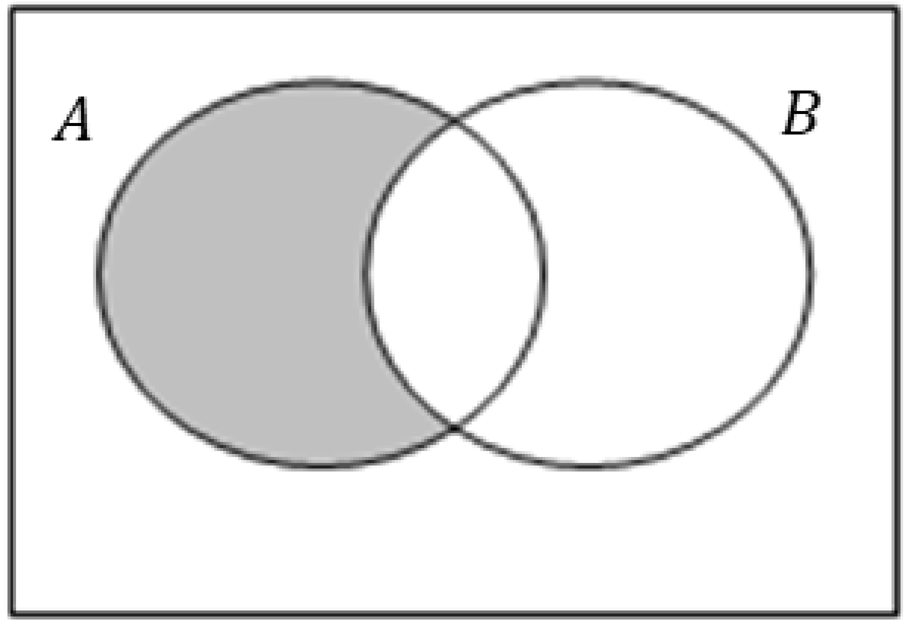


Example 12

If A = {1,2,3,4} and B = {5,7,9}
Then, A∩B = ∅

1.4.5 Difference / Subtraction sets

The difference between set A and B written as A\B or A-B (read as A minus B) is a set containing all elements in A but not in B.



- A-B⇔A∩B′
- A-B⇔A\B

Example 13

If A = {1,3,4,5} and B = {4,5,7,9,10}
A - B = {1,3}
B - A = { 7,9,10}
Here, we can see (A-B) ≠ (B-A)

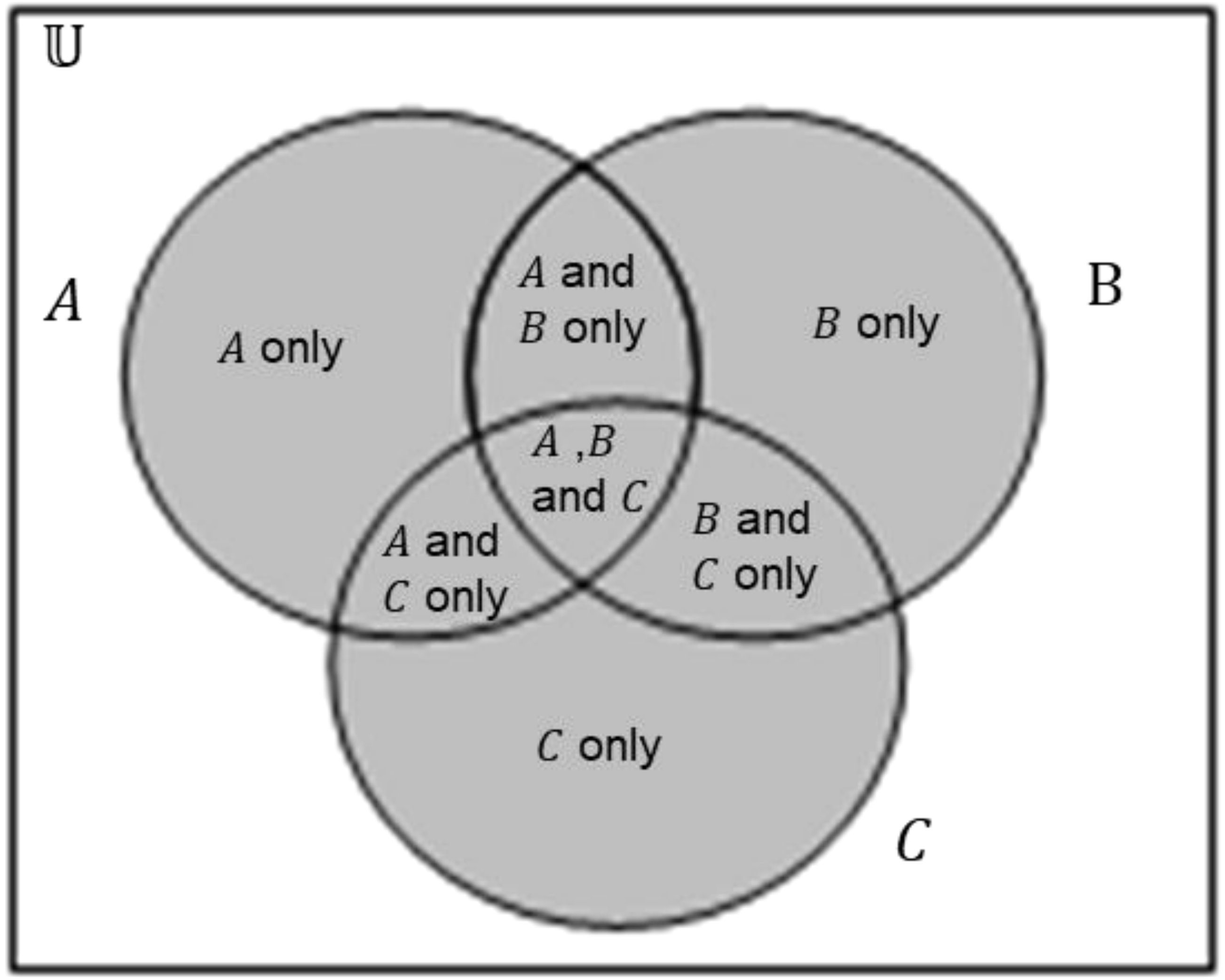
Exercise 3

1. Let U = {1,2,3,4,5,6,7,8}. If A = {2,4,6,8}, and B = {1,2,5}. Find:
   a. $A \cup B$
   b. $A \cap B$
   c. A-B
   d. B-A
   e. A′

2. If A = {1,3,4,5} and B = {7,9,10,11,14}. Find:
   a. $A \cup B$
   b. $A \cap B$
   c. A-B
   d. B-A

1.5 Venn Diagrams

A Venn Diagram is an illustration that uses overlapping circles to show the logical relationship between two or more sets of items. It makes math easier because it helps us see the whole situation at a glance. It is convenient to choose a larger set that contains all of the elements in all of the sets being considered. This larger set is called the universal set, and is usually given the symbol U.

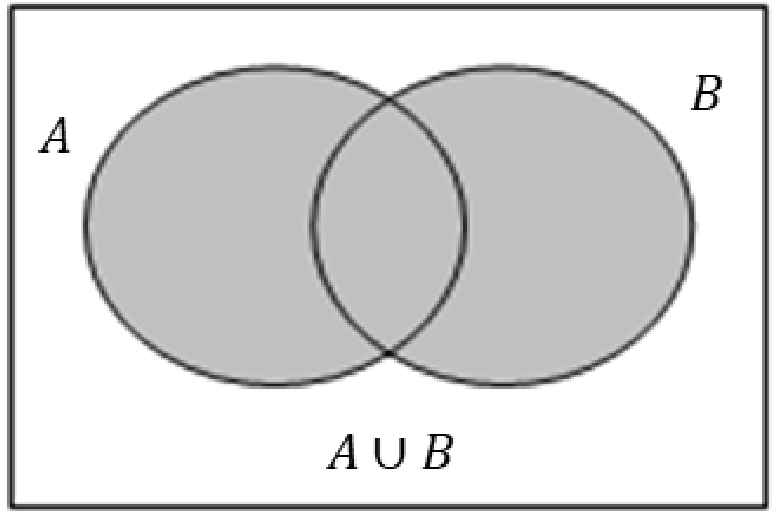


1.5.1 Operations of Sets

Number of Elements.

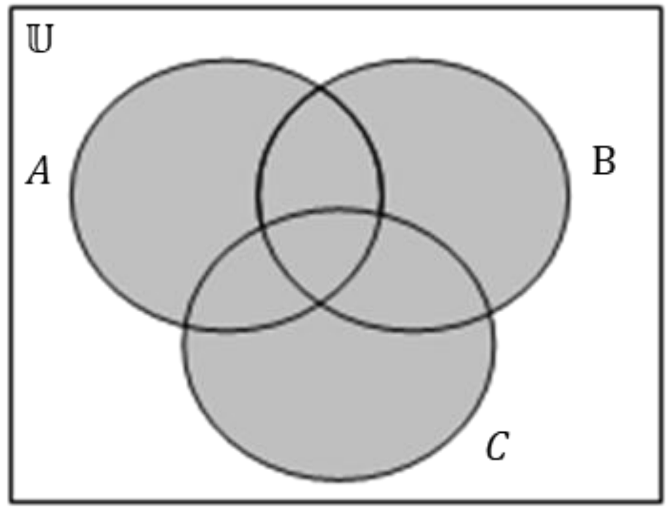
- Let A and B be two finite sets, then,

$$|A \cup B| = |A| + |B| - |A \cap B|$$



- Let A , B and C be three finite sets, then

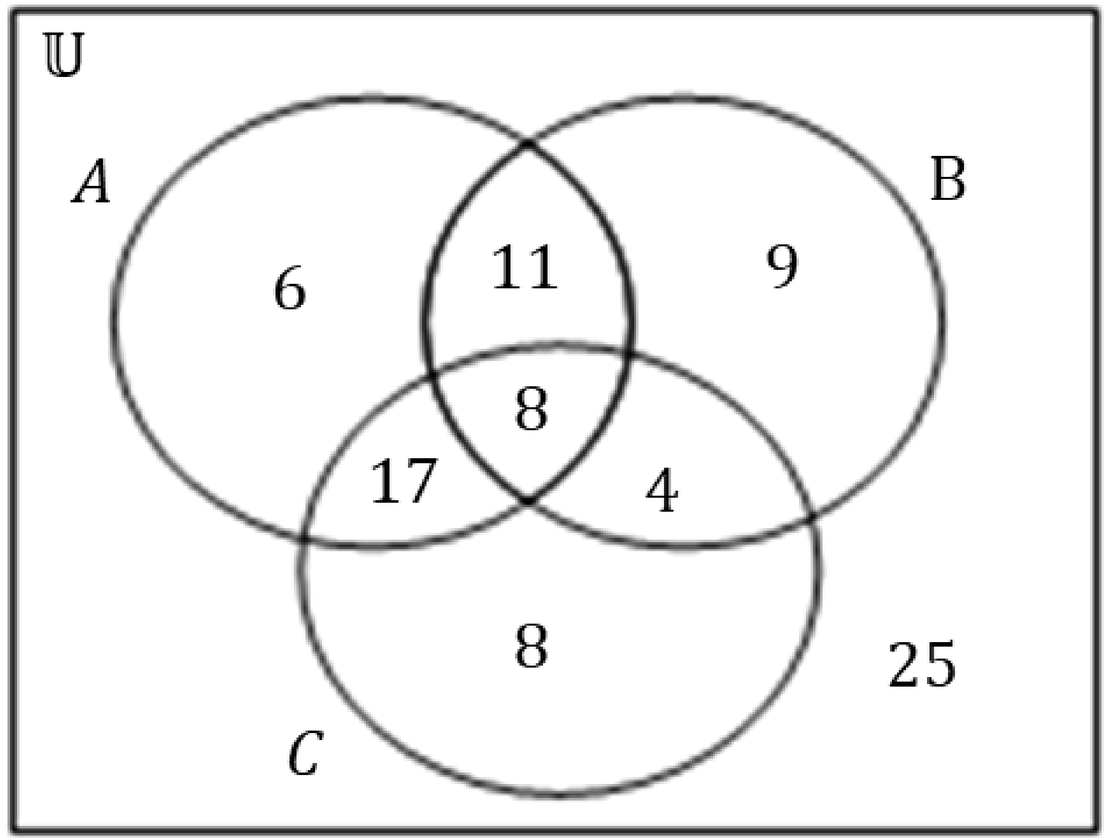
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



1.5.2 Use the data in the Venn diagram to answer questions

Example 14

a. How many elements are in set A?
b. How many elements are in set A only?
c. How many elements are in A or B or C?
d. How many elements are in none?
e. How many elements are in B but not C?
f. How many elements are in B and C only?



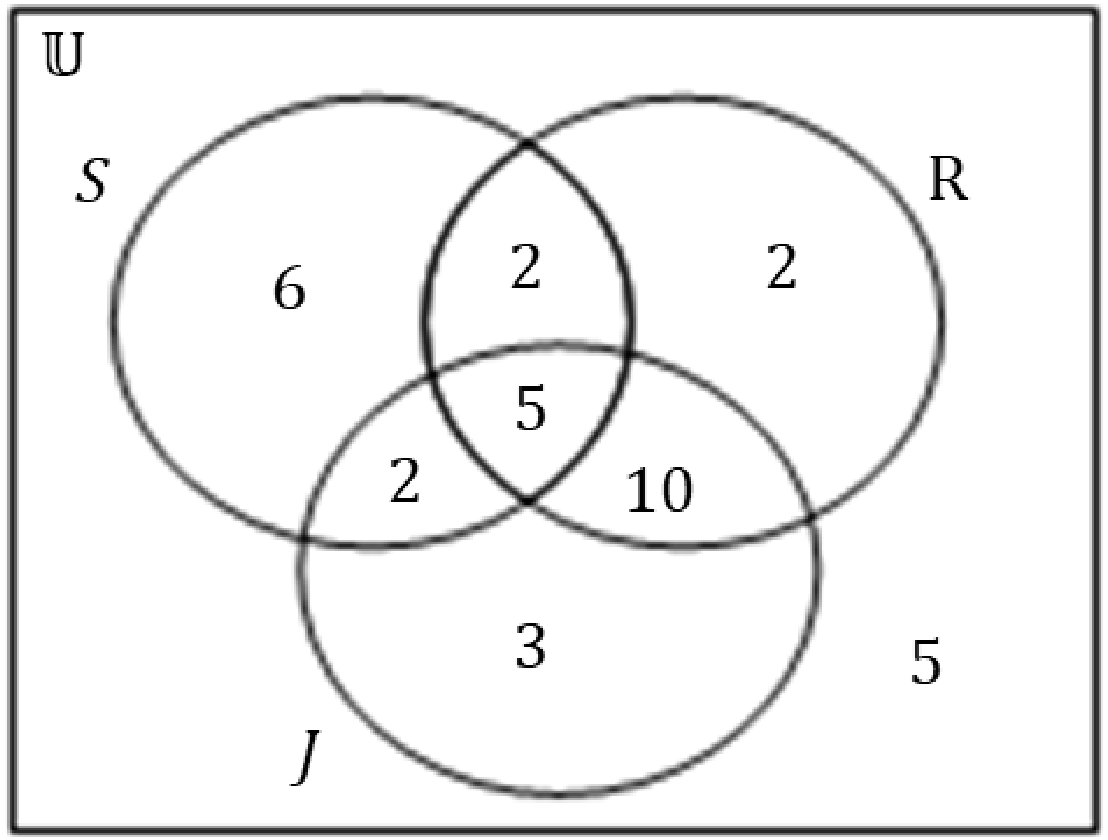
Solution

a. $|A| = 6+11+17+8 = 42$
b. $|A|$ only $= 6$
c. $|A \cup B \cup C| = 6+11+8+17+4+9+8 = 63$
d. $|A \cup B \cup C|' = 25$
e. $|B-C|$ or B only $= 9+11 =$ 20
f. $|B \cap C|$ only $= 4$

Exercise 4

A survey was carried out on 35 students at a college on what types of exercise they do on their weekends. The following result shows the number of students from the kind of their exercise. (Swimming(S), Running(R,and Jogging (J).

a. How many of the students are doing running and jogging only?
b. How many of the students are doing swimming and running?
c. How many of the students are doing swimming and running but not jogging?
d. How many students jog on the weekend?
e. How many students do all three exercises?
f. How many students don't do any exercise?



1.5.3 Solving Problem using Venn Diagram

Example 15

In a group of 100 customers at the Pizza Kampung,
80 of them ordered mushrooms topping,
72 of them ordered pepperoni topping and
60 customers ordered both mushrooms and pepperoni toppings on their pizza.

i. Draw a Venn Diagram to illustrate the data.
ii. How many customers ordered mushrooms but no pepperoni?
iii. How many customers ordered pepperoni but no mushrooms?
iv. How many customers ordered neither of these two toppings?

Solution

i. Draw a Venn Diagram to illustrate the data.

Step 1: Write down the elements in the intersection M∩P (all 2 items)

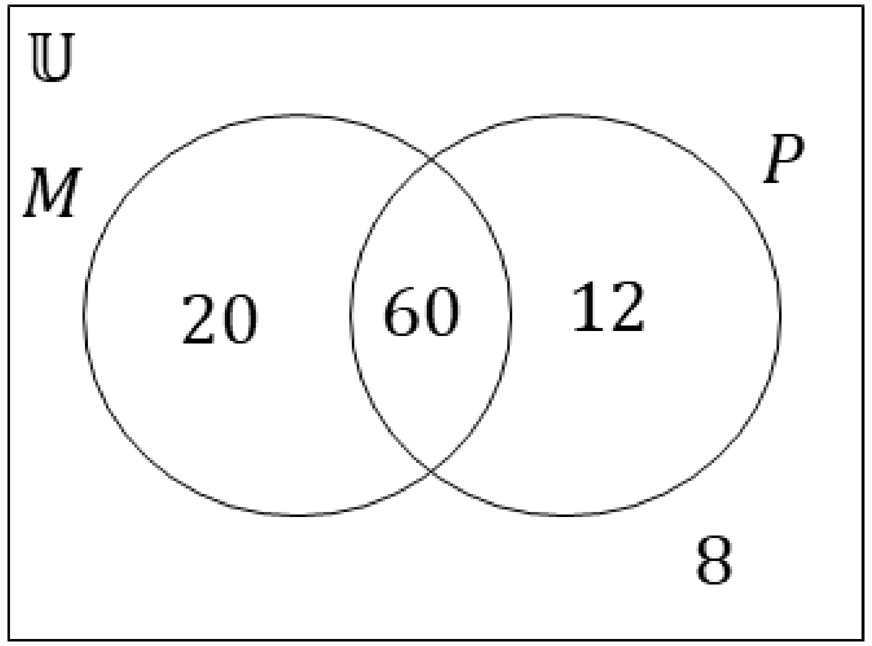
$M \cap P = 60$

Step 2: Write down the remaining elements in the respective sets and in Universal Set.

P only = 72-60 = 12

M only = 80-60 = 20

$(P \cup M)' = 100-20-60-12 = 8$



ii. How many customers ordered mushrooms but no pepperoni? = 20

iii. How many customers ordered pepperoni but no mushrooms? = 12

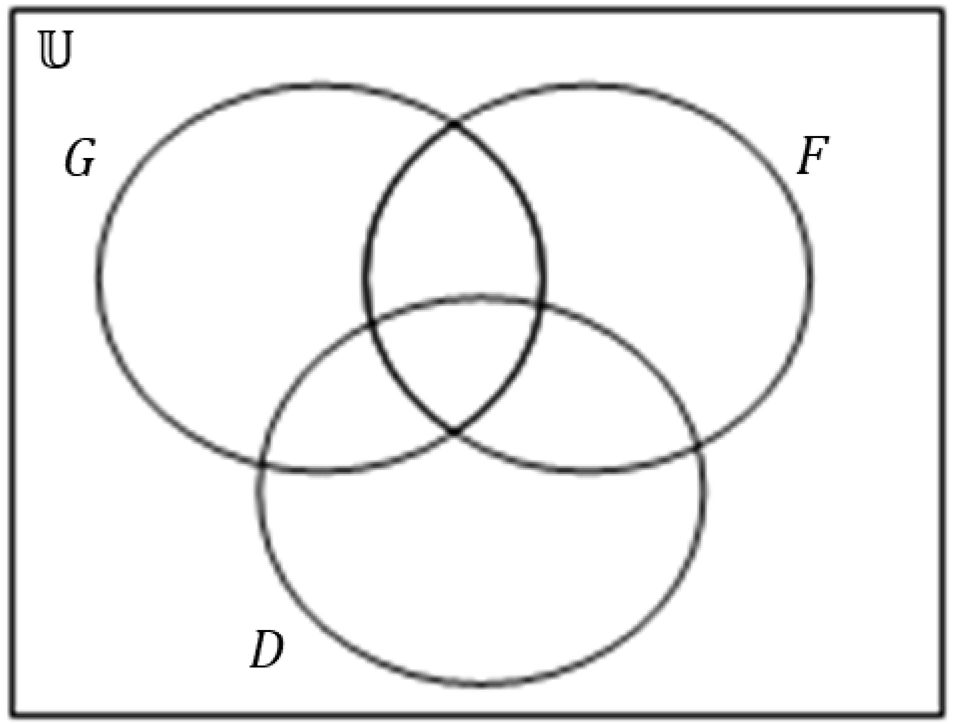
iv. How many customers ordered neither of these two toppings? = 8

Example 16

In May 2024, 350 customers came to Adam Supermarket to buy fresh milk.

Records show that:

- 74 customers bought Good Day and Farm Fresh
- 41 customers bought Good Day and Dutch Lady
- 33 customers bought Farm Fresh and Dutch Lady
- 165 customers bought Good Day
- 80 customers bought Dutch Lady
- 113 customers bought Farm Fresh
- 20 customers bought all three brands of healthy milk



i. Draw a Venn Diagram to illustrate the data.

   Step 1: Write down the elements in the intersection $G \cap F \cap D$ (all three brands)

   Step 2: Write down the elements in the intersections: $G \cap F$, $G \cap D$, and $F \cap D$.

   Step 3: Write down the elements in the respective sets and in the Universal Sets.

ii. How many customers did not choose to buy any of the above brands?

iii. How many customers buy Good Day and Farm Fresh but not Dutch Lady?

Solution

74 customers bought Good Day and Farm Fresh
41 customers bought Good Day and Dutch Lady
33 customers bought Farm Fresh and Dutch Lady
165 customers bought Good Day
80 customers bought Dutch Lady
113 customers bought Farm Fresh
20 customers bought all three brands of healthy milk

i. Draw a Venn Diagram to illustrate the data.

Step 1 : $G \cap F \cap D = 20$

Step 2 : $G \cap F$ only = 74-20 = 54

$G \cap D$ only = 41-20 = 21

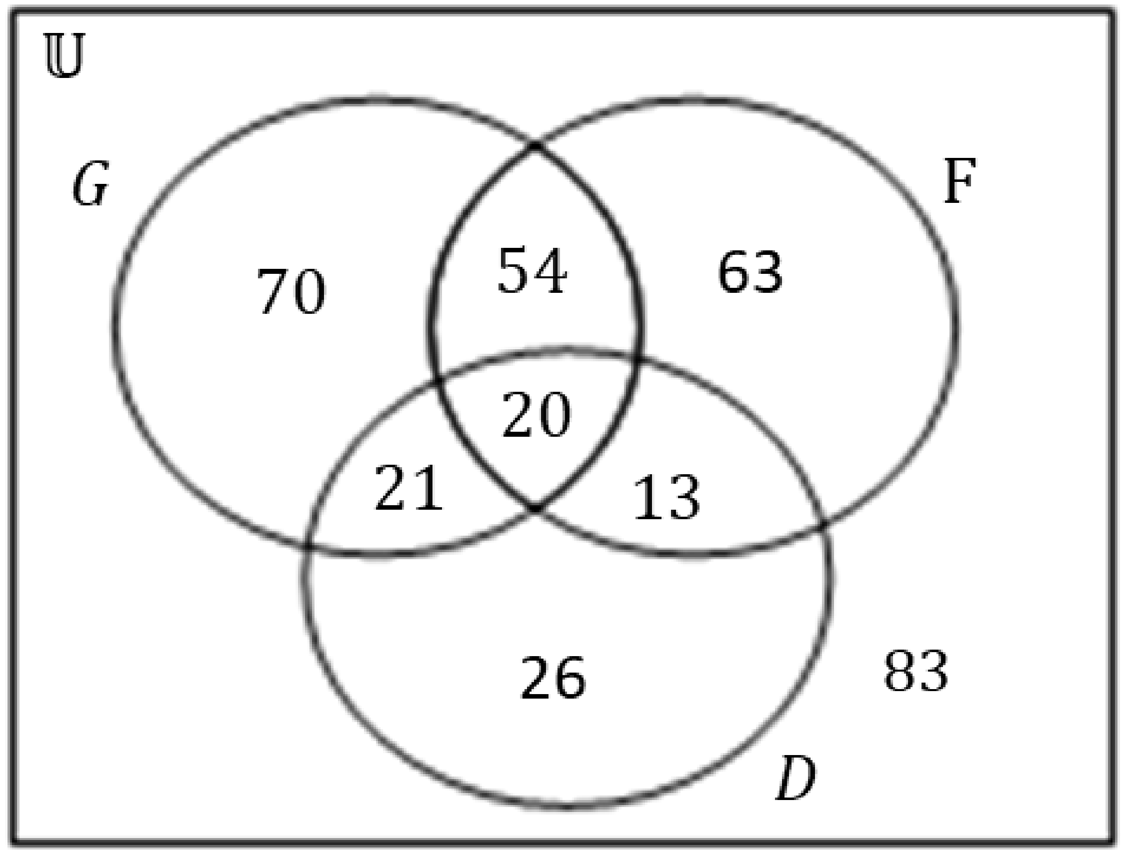
$F \cap D$ only = 33-20 = 13

Step 3 : G only = 165-54-20-21 = 70

D only = 80-21-20-13 = 26

F only = 150-54-20-13 = 63

$(G \cup F \cup D)'$ = 350-70-54-20-21-63-13-26 = 83



Example 17

A lecturer in the Mathematics Department takes a survey on the first day of class to determine how many students know about Calculus, Discrete Mathematics, and Statistics. The finding is that out of 50 students in the class.

30 know about Calculus

18 know about Discrete Mathematics

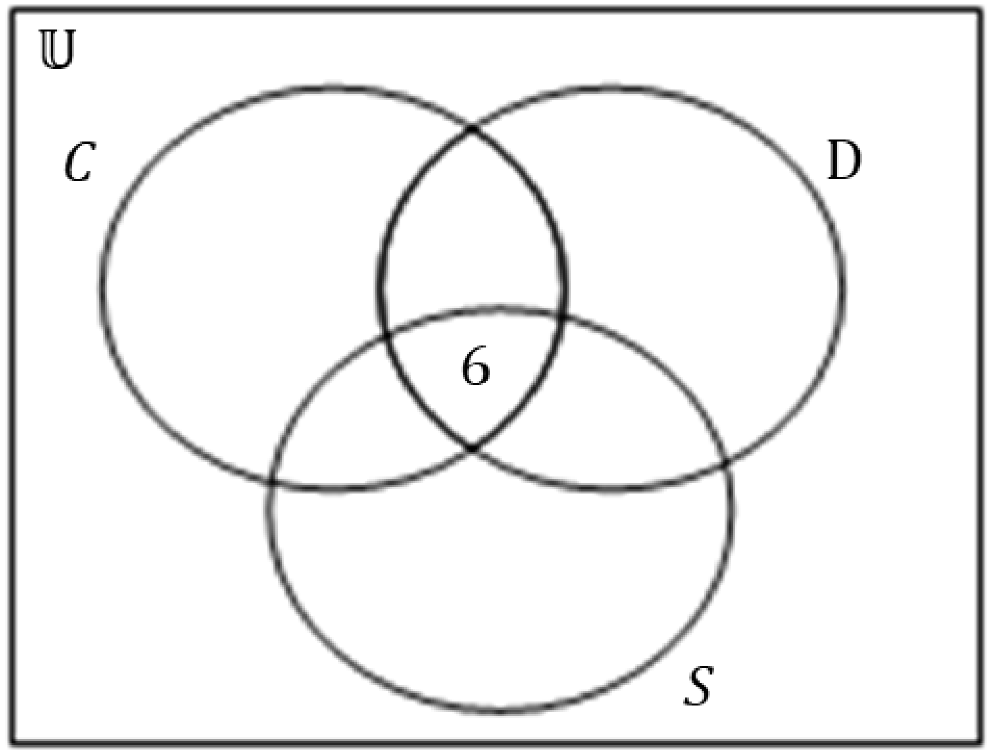
26 know about Statistics

9 know both Calculus and Discrete Mathematics

16 know both Calculus and Statistics

8 know both Discrete Mathematics and Statistics

47 know at least one of the subjects. =($C \cup D \cup S$)



i. How many students know all the three subjects?

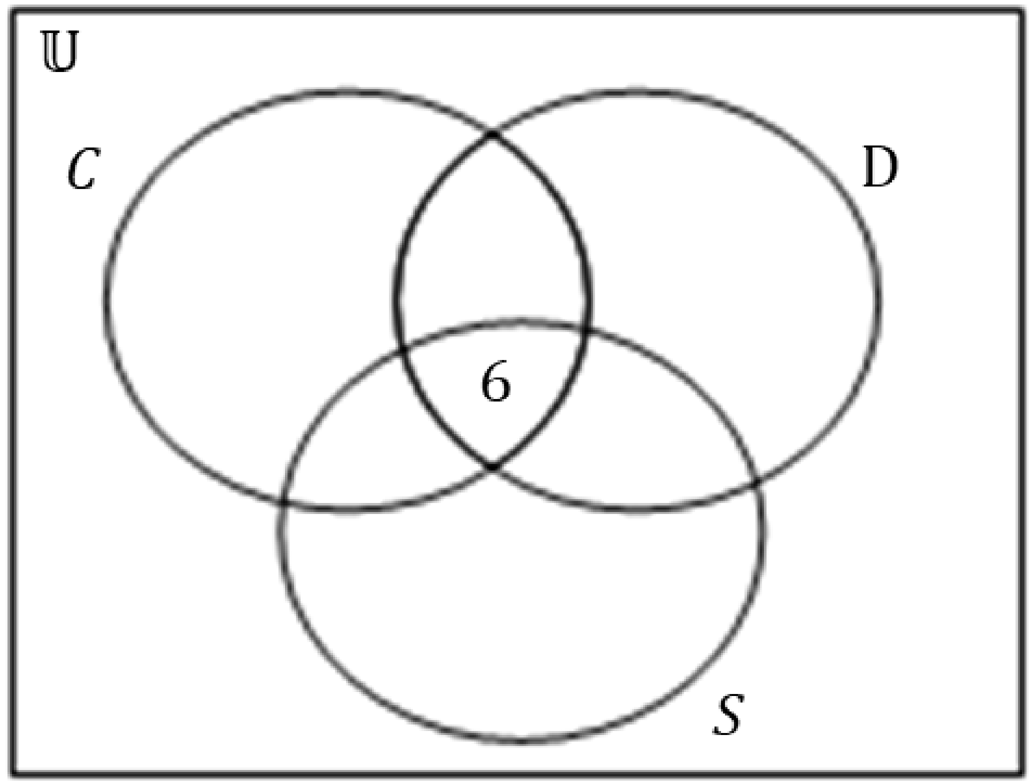
ii. Draw a Venn Diagram to illustrate the data

iii. How many students know none of the three subjects?

iv. How many students know Calculus only?

Solution

30 know about Calculus
18 know about Discrete Mathematics
26 know about Statistics
9 know both Calculus and Discrete Mathematics
16 know both Calculus and Statistics
8 know both Discrete Mathematics and Statistics
47 know at least one of the subjects. =(C∪D∪S)



i. How many students know all the three subjects?

Step 1: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$|C \cup D \cup S| = |C|+|D|+|S|-|C \cap D|-|C \cap S|-|D \cap S|+|C \cap D \cap S|$

$47 = 30+18+26-9-16-8+|C \cap D \cap S|$

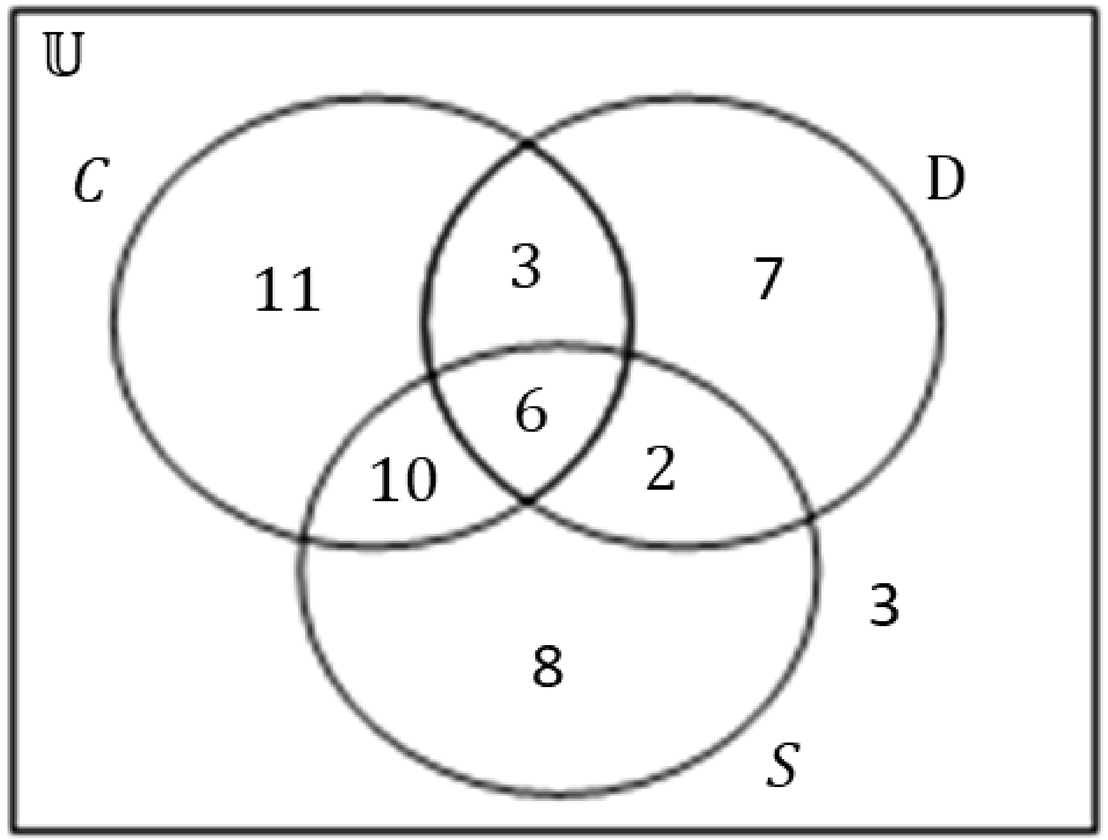
$47 = 41+|C \cap D \cap S|$

$47-41 = |C \cap D \cap S|$

$6 = |C \cap D \cap S|$

Solution (continue)

30 know about Calculus
18 know about Discrete Mathematics
26 know about Statistics
9 know both Calculus and Discrete Mathematics
16 know both Calculus and Statistics
8 know both Discrete Mathematics and Statistics
47 know at least one of the subjects. =(C∪D∪S)



Step 2 : D∩S only = 8-6 = 2
C∩S only = 16-6 = 10
C∩D only = 9-6 = 3

Step 3: S only = 26-10-6-2 = 8
D only = 18-6-2-3 = 7
C only = 30-10-6-3 = 11
(C∪D∪S)′= 50-47 = 3

ii. Draw a Venn Diagram to illustrate the data
iii. How many students know none of the three languages? = 3
iv. How many students know Calculus only? = 11

Exercise 5

1. 85 students were asked about their favorite breakfast, whether they like to eat nasi lemak (N), roti canai (R) or fried noodles (F). The results are shown below.

   35 students like nasi lemak
   37 students like roti canai
   26 students like fried noodles
   20 students like nasi lemak and roti canai
   14 students like nasi lemak and fried noodles
   3 students like roti canai and fried noodles
   2 students like all the meals.

i. Draw a Venn Diagram to illustrate the data
ii. How many of these students like nasi lemak and roti canai only.
iii. How many of these students liked roti canai and fried noodles but not nasi lemak

2. In a group of 60 students; 25 play table tennis, 16 play football, 22 play bowling, 8 play table tennis and football, 6 play bowling and football, 5 play table tennis and bowling and 2 students play all three.
   i. Draw a Venn Diagram to illustrate the data.
   ii. Find the number of students who play exactly one game.
   iii. How many students choose none of all the three sports.
   iv. Calculate the total number of students involved in at least two games.

3. A group of people were surveyed about their preferences for three types of music: Pop (P), Rock (R), and Jazz (J). 40 people enjoy Pop, 35 enjoy Rock, 25 enjoy Jazz, 15 enjoy both Pop and Rock, 10 enjoy both Rock and Jazz, 12 enjoy both Pop and Jazz and 5 enjoy all three genres.
   i. Draw a Venn Diagram to illustrate the data.
   ii. Calculate how many people enjoy only one genre of music

4. A lecturer in the Computer Science Department conducts a survey on the first day of class to see how many students are familiar with Programming, Data Structures, and Algorithms. The findings show that out of a total of 60 students:

11 know about Programming only
7 know about Data Structure only
7 know about Algorithms only
15 know both Programming and Data Structures
18 know both Programming and Algorithms
12 know both Data structure and Algorithms
9 know all the three subjects.

i. Draw a Venn Diagram to illustrate the data.
ii. Calculate how many students aren’t familiar with all three subjects.

Thank You!

With the hope that all the learning outcomes below have been achieved.

1. Understand the basic concept of set theory.
2. Perform operations on sets, including union, intersection, difference, and complement.
3. Use Venn diagrams to visualize and solve problems related to sets and their relationships.

Topic 2
Functions

2.1 Introduction

2.2 Types of Functions

- 2.2.1 Injective
- 2.2.2 Surjective
- 2.2.3 Bijective

2.3 Inverse Function

Learning Outcome

By the end of this topic, student should be able to:

1. Explain the key components of a function, including domain, codomain, and range, image and object.
2. Define and explain the concepts of injective (one-to-one), surjective (onto), and bijective functions.
3. Find the the inverse of a function.

2.1

Introduction

What are functions?

- Functions are a special kind of relation that mapping from a set of inputs A (called the **domain**) to a set of possible outputs B(called the **codomain**) such that **all element** in the domain must have **only 1 image** in codomain.
- It denoted as $f: A \to B$
- $f(x) = y$ means the function f maps x to y.
- A function can be represented by using **arrow diagram**, **ordered pairs** and a **graph.**

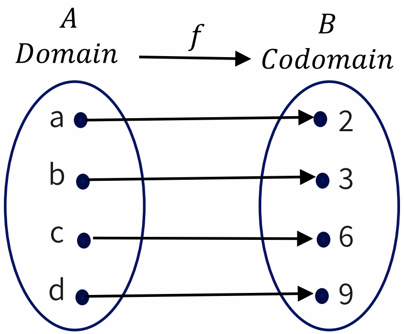


Figure 1: Arrow Diagrams

Ordered pairs

$f = \{(a,2),(b,3),(c,6),(d,9)\}$

- Compare the four relations on the figures below. Which of the following relations are functions?

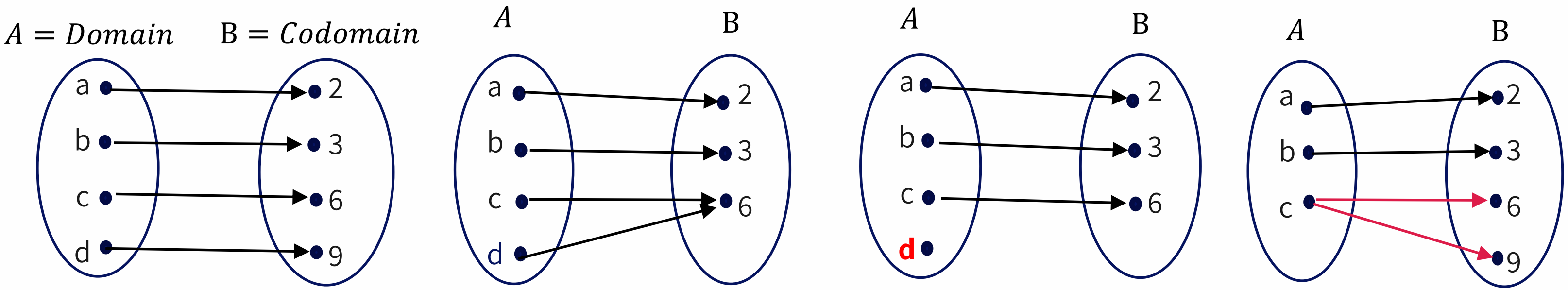
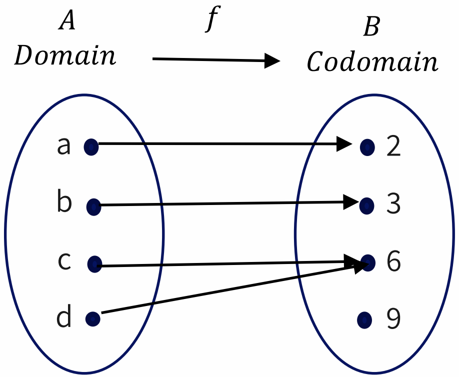


Figure 2: Relation 1 Figure 3: Relation 2 Figure 4: Relation 3 Figure 5: Relation 4

- **Relation 1** – All element in $domain$ has only one image in $codomain$. This relation is a **functions.**
- **Relation 2** – All element in $domain$ has only one image in $codomain$, This relation is a **functions.** although there are 2 elements in $domain$ map to the same element in $codomain$.
- **Relation 3** – There is 1 element in the domain doesn't have an image in the codomain. This relation is **not functions.**
- **Relation 4** – There is 1 element in the domain has 2 images in the codomain, This relation is **not functions.**

Terminology in functions?

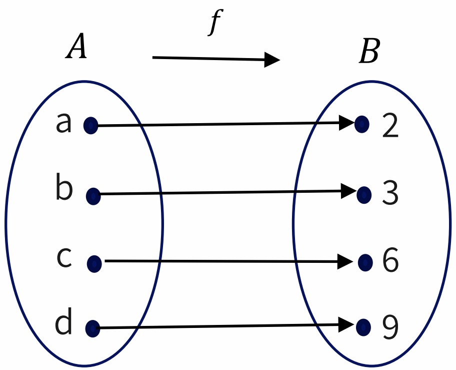


The diagram above show the relation that maps elements of set A to the elements in Set B.

- **Domain**: The set of all possible inputs (e.g. $A = \{a, b, c, d\}$
- **Objects**: An element in the domain that maps to a specific element in the codomain. (e.g. the object of **2** is **a**).
- **Codomain**: The set of all possible inputs (e.g. $B = \{2,3,6,9\}$
- **Range**: The set of actual outputs produced by the function. (e.g. Range $= \{2,3,6\}$
- **Image**: The outputs of $f(x)$ for a specific input x (eg., the image of **a** is **2**)

Example 1

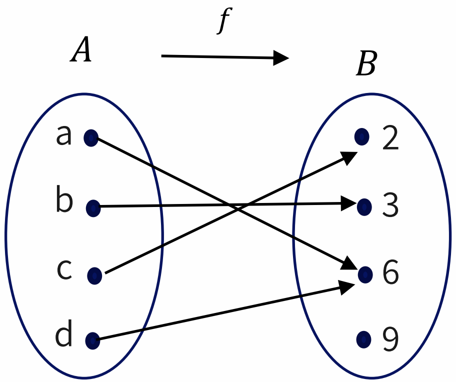
Let's say, Set $A = \{a, b, c, d\}$ and B $= \{2,3,6,9\}$ where A is the set of characters and B is the set of integers. Then given an ordered pairs of function $f = \{(a, 2), (b, 3), (c, 6), (d, 9)\}$. The arrow diagram is shown below.



- The domain is $\{a, b, c, d\}$
- The codomain is $\{2,3,6,9\}$
- The Range is $\{2,3,6,9\}$
- The image of a is 2
- The object of 3 is b

Example 2

Let's say, Set $A = \{a, b, c, d\}$ and $\mathrm{B} = \{2,3,6,9\}$ where A is the set of characters and B is the set of integers. Then given an ordered pairs of function $f = \{(a, 6), (b, 3), (c, 2), (d, 6)\}$. The arrow diagram is shown below.



- The domain is $\{a, b, c, d\}$
- The codomain is $\{2,3,6,9\}$
- The Range is $\{2,3,6\}$
- The image of **a** is **6**
- The object of **6** is **a** and **d**

Example 3

Given that $f(x) = 3x + 5$ with the domain $\{2,4,5\}$. Find:

a. the range of the function f.
b. the image of 6
c. $f(-5)$
d. $f(9)$
e. the object of 26
f. the object of 35

Example 3 -Solution

Given that $f(x) = 3x + 5$ with the domain $\{2,4,5\}$. Find:

a. the range of the function f.

$f(2) = 3(2) + 5 = 11$

$f(4) = 3(4) + 5 = 17$

$f(5) = 3(5) + 5 = 20$

$\therefore$ range of the function $f = \{11,17,20\}$.

b. the image of 6

$f(6) = 3(6) + 5 = 23$ #

c. $f(-5) = 3(-5) + 5 = -10$ #

d. $f(9) = 3(9) + 5 = 32$ #

e. the object of 26

$$f(x) = 3x + 5$$
$$26 = 3x + 5$$
$$26 - 5 = 3x$$
$$21 = 3x$$
$$\frac{21}{3} = x$$
$$7 = x$$ #

f. the object of 35

Exercise 1

1. Show that whether the relations below is a functions or not.
   Set $A = \{10,6,15,12\}$ to $B = \{x, y, z\}$

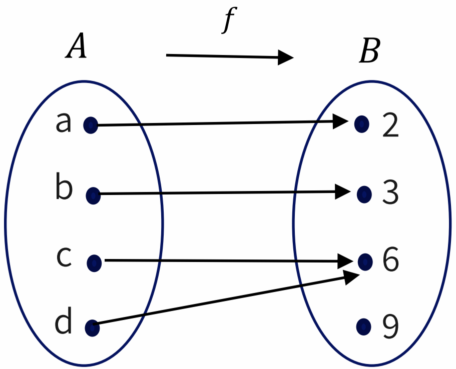
   a. Relation 1 $= \{(10, x), (6, y), (15, z), (12, y)\}$
   b. Relation 2 $= \{(10, x), (6, y), (15, z)\ \}$
   c. Relation 3 $= \{(10, x), (10, y), (15, z), (12, y)(6, z)\}$

2. Which relations below are functions

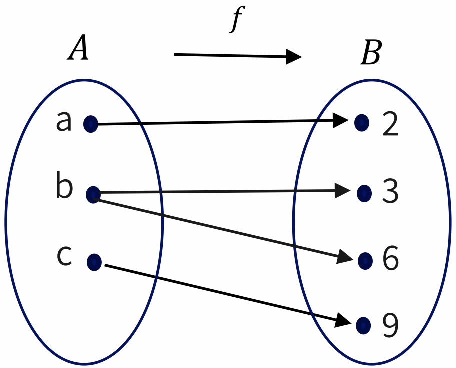
   a. Relation 1 $= \{(1,2), (-4,5), (-1,2), (8, -16)\}$
   b. Relation 2 $= \{(10,12), (11,15), (15,17)\}$
   c. Relation 3 $= \{(4,16), (4,20), (6,36), (7,42)(8,60)\}$

3. Determine which relations below are functions.

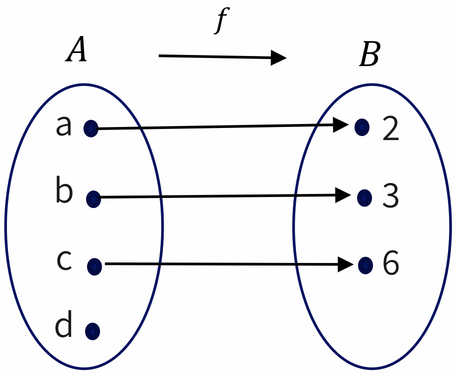
a.



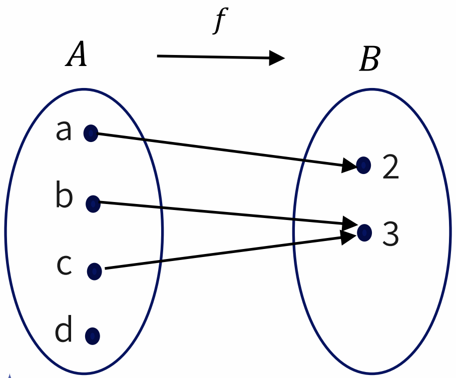
b.



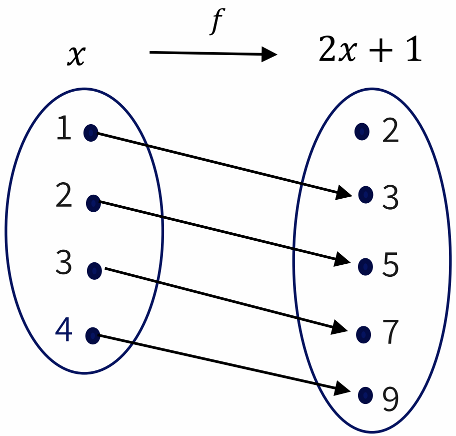
c.



d.



4. The diagram below show the function of $f(x) = 2x + 1$. Determine:



a. the domain of the function f
b. the range of the function f
c. the object of 9
d. the image of 2

5. Draw arrow diagrams for the following functions with the domain $\{1,2,3\}$. Then state the range of each function.

a. $f(x) = 3x - 2$
b. $f(x) = x^2$
c. $f(x) = x^2 + 1$

6. Given that $f(x) = 4x - 2$. Find:

a. the image of 3
b. $f(4)$
c. $f(-5)$
d. $f(9)$
e. the object of 18

2.2

Types of function

2.2.1 Injective
2.2.2 Surjective
2.2.3 Bijective

2.2.1 Injective (one to one) function

Let $f: A \to B$ is a function . **Injective** or one to one function define that each element of Set A is mapped with an unique element of Set B.

Figure 6 shows 2 examples of Injective function.

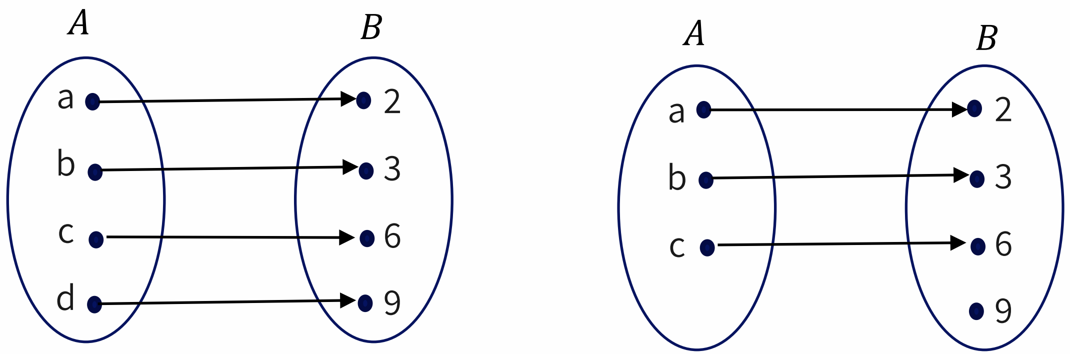


Figure 6: The Injective function

2.2.2 Surjective (onto) function

Let $f: A \to B$ is a function. **Surjective** or onto function define that **every element** of **Set B** is mapped with at least one element of Set A.
Figure 7 and 8 shows an examples of surjective and not surjective function.

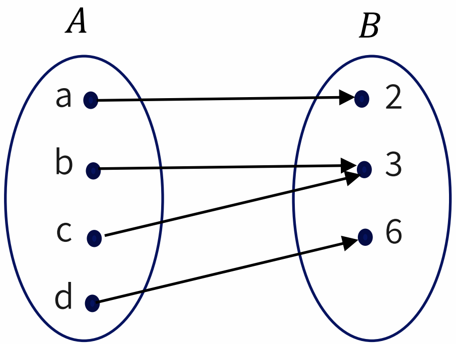


Figure 7: The Surjective function

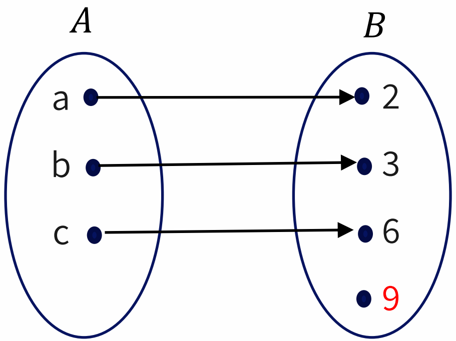


Figure 8: Not Surjective function

2.2.3 Bijective

Let $f: A \to B$ is a function. **Bijective** if it is injective and surjective.
Every element of Set B is exactly mapped with one element of Set A.
Figure 9 show an examples of bijective function.

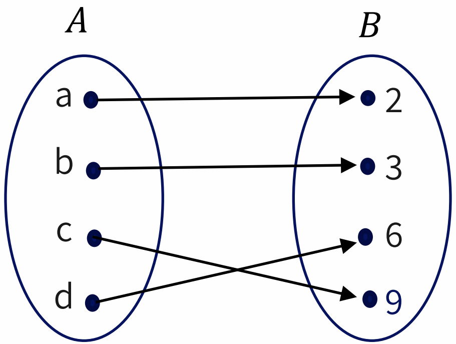
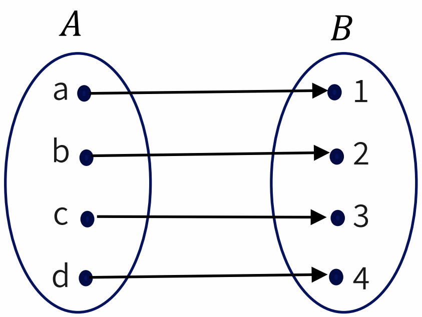


Figure 9: The Bijective function

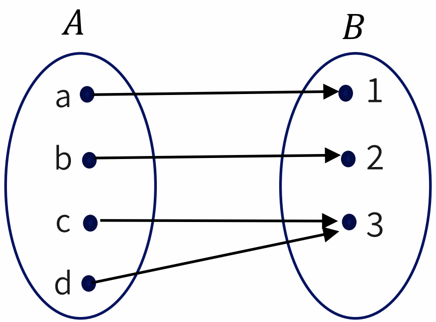
Exercise 2

1. Determine whether diagram below injective, surjective or bijective function.

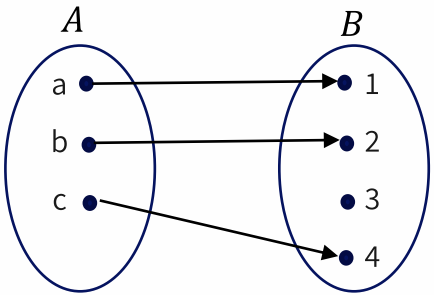
a.



b.



c.



2. Let $f: X = \{5,10,15,29\}$, and $Y = \{p, q, r\}$, where $f = \{(5, p), (10, q), (15, r)(29, r)\}$. Determine whether f is injective, surjective, or bijective.

3. Let $h: M = \{x, y, z\}$, and $N = \{m, n, p\}$, where $h = \{(x, p), (y, m), (z, n)\}$. Determine whether h is injective, surjective, or bijective.

4. Let $f: S = \{3,6,9\}$, and $T = \{5,8,11,13\}$, where $f = \{(3,5), (6,11), (9, 8)\}$. Determine whether f is injective, surjective, or bijective.

5. Let $j: A = \{-1,0,1\}$, and $B = \{r, s, t\}$, where $j = \{(-1, r), (0, s), (1, s)\}$. Determine whether j is injective, surjective, or bijective.

2.3

Inverse function

If $f(x)$ is one to one function, such that $f^{-1}(x)$ is $f^{-1}(y) = x$ if and only if $f(x) = y$.

<u>Step for inverse function:</u>

Step 1 : Let $y = f(x)$
Step 2: Make x as a subject $\Rightarrow \therefore x = f^{-1}(y)$
Step 3: Change all the variable y to x

Example 4

Find the inverse function for $f(x) = 5x + 3$

Step 1 : Let $y = f(x)$

$$y = 5x + 3$$

Step 2: Make x as a subject

$$y - 3 = 5x$$

$$\frac{y-3}{5} = x$$

$$\therefore x = f^{-1}(y) = \frac{y-3}{5}$$

Step 3: Change all the variable y to x

$$\therefore f^{-1}(x) = \frac{x-3}{5}$$

<u>Step for inverse function:</u>

Step 1 : Let $y = f(x)$
Step 2: Make x as a subject $\Rightarrow \therefore x = f^{-1}(y)$
Step 3: Change all the variable y to x

Example 5

Given that $f(x) = -4 + 2x$, determine the:

a. Inverse function $f^{-1}(x)$

b. $f^{-1}(6)$

a. Step 1 : Let $y = f(x)$

$$y = -4 + 2x$$

Step 2: Make x as a subject

$$y + 4 = 2x$$

$$\frac{y+4}{2} = x$$

$$\therefore x = f^{-1}(y) = \frac{y+4}{2}$$

Step 3: Change all the variable y to x

$$\therefore f^{-1}(x) = \frac{x+4}{2}$$

Step for inverse function:

Step 1 : Let $y = f(x)$

Step 2: Make x as a subject $\Rightarrow \therefore x = f^{-1}(y)$

Step 3: Change all the variable y to x

$b.\ f^{-1}(x) = \dfrac{x+4}{2}$

$$f^{-1}(6) = \frac{6+4}{2} = 5$$

Exercise 3

1. Find the inverse function $f^{-1}(x)$ for the following function.

   a. $f(x) = 3x + 4$

   b. $f(x) = 9x - 5$

   c. $f(x) = \frac{1}{2x}$

   d. $f(x) = x^2 - 6$

   e. $f(x) = x^2 + 7$

2. Let $g(x) = 3x + 5$, where $g\colon \mathbb{R} \to \mathbb{R}$.

   a. Find the inverse function $g^{-1}(x)$

   b. Calculate $g^{-1}(11)$

3. Let $f(x) = \frac{2x-1}{3}$, where $f\colon \mathbb{R} \to \mathbb{R}$.

   a. Find the inverse function $f^{-1}(x)$

   b. Calculate $f^{-1}(5)$

Thank You
&
Good Luck

Topic 3
Boolean Algebra

3.1 Introduction
3.2 Boolean Algebra Operations
3.3 Boolean Algebra Truth Table
3.4 Boolean Identities

Learning Outcome

By the end of this topic, student should be able to:

1. Explain the fundamental concepts and operations of Boolean Algebra, including AND, OR, NOT,
2. Recognize and interpret Boolean expressions and their equivalent truth tables.
3. Simplify complex Boolean expressions using Boolean laws, rules, and identities

3.1

Introduction

- **Boolean Algebra** is a branch of mathematics that deals with binary variables and logical operations.
- It is used in the design of electric circuits which is widely used in computer science, electrical engineering, and related fields.
- **Binary Variables**:
  Variables in Boolean Algebra take only two possible values:
  1 (True) or **0 (False)**.

3.2

Boolean Algebra Operations

3.2.1 Conjunction (AND) Operations
3.2.2 Disjunction (OR) Operations
3.2.3 Negation (NOT) Operations

Boolean Algebra Operations

- There are 3 Boolean algebra operations that will always be used, which are Complement, Boolean Sum, and Boolean Product.
- Below is the table defining the symbols for all three basic operations.

Boolean Algebra	Symbol	The values of Boolean algebra
Conjunction (And)	$\cdot$	$1 \cdot 1 = 1$ $1 \cdot 0 = 0$ $0 \cdot 1 = 0$ $0 \cdot 0 = 0$
Disjunction (Or)	$+$	$1 + 1 = 1$ $1 + 0 = 1$ $0 + 1 = 1$ $0 + 0 = 0$
Negation (Not)	$'$ or $\overline{}$	$\overline{0} = 1$ $\overline{1} = 0$

Example 1

Solve the Boolean algebraic below:

a. $1 \cdot 1 + \overline{(0 + 1)}$

b. $(1 \cdot 1) + (\overline{0 \cdot 1} + 0)$

c. $(\bar{1} \cdot \bar{0}) + (1 \cdot 0)$

Solution:

a. $1 \cdot 1 + \overline{(0 + 1)}$

$= 1 + \overline{(1)}$

$= 1 + 0$

$= 1$

3.3

Boolean Algebra Truth Table

These tables show the output of a Boolean expression for all possible combinations of input values.

1. Conjuction (And) $(\cdot)$

x	y	$x \cdot y = (xy)$
1	1	1
1	0	0
0	1	0
0	0	0

2. Disjunction (Or) $(+)$

p	q	$x + y$
1	1	1
1	0	1
0	1	1
0	0	0

3. Negation (Not) $(\,'$ or $\overline{}\,)$

x	$\bar{x}$
1	0
0	1

Example 3

Find the truth table of the Boolean function represented by $F(x, y) = (x \cdot \bar{y}) + y$

x	y	$\bar{y}$	$(x \cdot \bar{y})$	$(x \cdot \bar{y}) + y$
1	1	0	0	1
1	0	1	1	1
0	1	0	0	1
0	0	1	0	0

Example 4

Find the truth table of the Boolean function represented by $F(x, y, z) = x + yz$

x	y	z	yz	$x + yz$
1	1	1	1	1
1	1	0	0	1
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

Exercise 1

Construct a truth table for the following Boolean functions:

a. $F(x, y) = x \cdot (y + \bar{x})$

b. $F(x, y) = (x + y) \cdot (\bar{x} \cdot \bar{y})$

c. $F(x, y) = (x \cdot \bar{x}) + [(y + \bar{y}) \cdot x]$

d. $F(x, y, z) = (x \cdot \bar{y}) + y \cdot (\bar{x} + z)$

3.4

Boolean Identities

- **Boolean identities** are fundamental rules or properties used to simplify the design of circuits
- Each of the identities can be proven using a truth table.

Boolean Identities		
No.	**Name**	**Identity**
1.	Double Complement Law	$\bar{\bar{x}} = x$
2.	Complement Law	$x + \bar{x} = 1$ $x \cdot \bar{x} = 0$
3.	Idempotent Law	$x + x = x$ $x \cdot x = x$
4.	Identity Law	$x + 0 = x$ $x \cdot 1 = x$
5.	Dominance Law	$x + 1 = 1$ $x \cdot 0 = 0$

Boolean Identities		
No.	**Name**	**Identity**
6.	Commutative Law	$x + y = y + x$ $xy = yx$
7.	Associative Law	$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$
8.	Absorption Law	$x + xy = x$ $x(x + y) = x$
9.	Distributive Law	$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$
10.	De Morgan's Law	$(\overline{xy}) = \bar{x} + \bar{y}$ $(\overline{x + y}) = \bar{x}\bar{y}$

Table 1: Boolean Identities

Example 5

Prove the absorption law $x + xy = x$ using the other laws in Table 1.

Solution

$x + xy = x\,(1 + y)$ Distributive Law
$= x\,(1)$ Dominance Law
$= x$ Identity Law

Example 6

Use the Boolean Identities to simplify the Boolean expression of $(x + y) \cdot \overline{(\bar{x} \cdot y)}$.

Solution

$$
\begin{aligned}
(x + y) \cdot \overline{(\bar{x} \cdot y)} &= (x + y) \cdot (\bar{\bar{x}} + \bar{y}) && \text{De Morgan's Law} \\
&= (x + y) \cdot (x + \bar{y}) && \text{Double Negation} \\
&= x + (y \cdot \bar{y}) && \text{Distributive Law} \\
&= x + 0 && \text{Complement Law} \\
&= x && \text{Identity Law}
\end{aligned}
$$

Example 7

Use the Boolean Identities law to simplify the following Boolean expression $(x + y) \cdot (\bar{x} \cdot \bar{y})$

Solution

$(x + y) \cdot (\bar{x} \cdot \bar{y})$	$= [(x + y) \cdot \bar{x}] \cdot \bar{y}$	Associative Law
	$= [(x \cdot \bar{x}) + (y \cdot \bar{x})] \cdot \bar{y}$	Distributive Law
	$= [0 + (y \cdot \bar{x})] \cdot \bar{y}$	Complement Law
	$= (y \cdot \bar{x}) \cdot \bar{y}$	Identity Law
	$= (\bar{x} \cdot y) \cdot \bar{y}$	Commutative Law
	$= \bar{x} \cdot (y \cdot \bar{y})$	Associative Law
	$= \bar{x} \cdot 0$	Complement Law
	$= 0$	Dominance Law

Exercise 2

1. Prove the absorption law $x(x + y) = x$ using the other laws in Boolean Identities.

2. Use the Boolean Identities law to simplify the following Boolean expressions.

   a. $\overline{(\bar{x} \cdot y)} + y$
   b. $(x \cdot y) + \bar{y}$
   c. $x\,(\bar{x} + y)$
   d. $(x \cdot y) + (x \cdot \bar{y})$
   e. $(x + \bar{y}) + (x + y)$
   f. $(x + y) + (\bar{x} \cdot \bar{y})$
   g. $(\bar{x} \cdot y) + x(x + y)$
   h. $[(x + y) + \mathrm{z}\,] \cdot x$

Thank You!

Topic 4
Algebraic Systems

Learning Outcome

By the end of this topic, student should be able to:

1. Identify and classify algebraic structures such as semigroups, monoids, groups and abelian groups.
2. State and explain common properties of operations in algebraic systems.
3. Apply criteria to determine whether a given set and operation form an algebraic system.

4.1 Introduction

- In mathematics, an algebraic structure is a set equipped with one or more operations that satisfy specific axioms. These structures are studied in abstract algebra and are foundational in many areas of mathematics.

- Examples of algebraic structures include semigroups, monoids, groups and abelian groups.

- Algebraic structures play a crucial role in computer science. Their properties and operations provide a framework for solving computational problems, designing algorithms, and developing secure systems.

Algebraic System

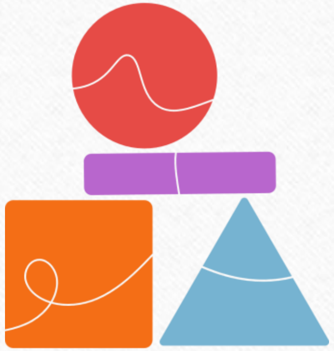
Algebraic systems are mathematical structures that consist of a set of elements ($Set\ A$) combined with one or more binary operations that satisfy specific properties or rules.

Example 1:

$(N, +)$, $(Z, +, -)$, $(R, +, \dots, -)$

Binary Operation

A **binary operation** is a rule or function that combines two elements from a set to produce a third element in the same set. It is called "binary" because it involves two inputs.



4.2 Operations

Closure, Associativity, identity and **Inverse** are fundamental properties often required for algebraic operations in Mathematical structures like groups, rings and field.

1. Closure

An operation $*$ on a set A has closure if, for every pair of elements $a, b \in A$, the result of a $* \, b$ is also in A.

$\therefore$ If $a, b \in A$ then $a * b \in A$

Example 2:

1. Integers (Z) closed under addition because adding two integers produces another integer. $(2 + 3 = 5)$.
2. The set of positive Integers (Z^+) is not closed under subtraction because $4 - 7 = -3$, which 3 and 5 are positive integers but -3 not positive integers.

2. Associativity

An operation $*$ on a set A is associative if the grouping of operations does not affect the result.

$\therefore$ If $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$,

Example 3:

1. Addition (+) of real number. $(2 + 3) + 5 = 2 + (3 + 5) = 10.$
2. Multiplication (·) of real number. $(2 \cdot 3) \cdot 5 = 2 \cdot (3 \cdot 5) = 30.$
3. Subtraction $(-)$ is not associative because $(2 - 3) - 5 \neq 2 - (3 - 5).$

3. Identity

An operation $*$ on a set A has an identity element e if, for every element $a \in A$, the operations of a with e leaves a unchanged.

$\therefore$ If $a * e = e * a$ for all $a \in A$,

Example 4:

1. Addition (+): The identity elements is 0 because $a + 0 = 0 + a = a$
2. Multiplication (·): The identity elements is 1 because $a \cdot 1 = 1 \cdot a = a$
3. Subtraction $(-)$: There is no identity element.

4. Inverse

An element a in a set A has an inverse b with respect to an operation $*$ if $a * b$ equals the identity.

$\therefore$ If $a * b = b * a = e$ for all $a, b \in A$,

Example 5:

1. Addition (+) of integers: The inverse of a is $-a$,
   Since $a + (-a) = 0$ (Identity element for addition)
2. Multiplication (·) of integers : The inverse of a is $\frac{1}{a}$,
   Since $a \cdot \left(\frac{1}{a}\right) = 1$ (Identity element for multiplication)

4.3 Semigroups

A **semigroup** is a basic algebraic structure that satisfies the following properties:

- Closure : The binary operation $*$ on a set A ensure that for any $a, b \in A$, then $a * b \in A$.
- Associativity: The operation $*$ satisfies $a * b) * c = a * (b * c)$ for all $a, b, c \in A$,

Monoid

A **monoid** is a semigroup that also includes an **identity element**. It satisfies the following properties:

- Closure: The binary operation $*$ is closed on A.
- Associativity: The operation $*$ is associative.
- Identity Element: There exists an element $a \in A$ such $a * e = e * a$ for all $a \in A$

Example 6

Prove that $(N, +)$ is semigroups.

Solution

Closure : For any $a, b \in N$, then $a + b \in N$.

e.g.: $3 + 4 = 7,$ since $3,4,7 \in N$

Associativity: $(a * b) * c = a * (b * c)$ for all $a, b, c \in N,$

e.g.: $(2 + 4) + 3 = 2 + (4 + 3) = 9,$ since $2,3,4 \in N$

Hence,$(N, +)$ is semigroups.

Example 7

Show that the set N is a monoid under multiplication.

Solution

Here, $N = \{1,2,3,4, \dots\}$

Closure : For any $a, b \in N$, then $a \cdot b \in N$.

We know that product of two natural numbers will get a natural number.

$\therefore$ Multiplication is a closed operation.

Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in N$,

$\therefore$ Multiplication of natural numbers is associative.

Identity: We have, $1 \in N$ such that

$a \cdot 1 = 1 \cdot a = a$ for all $a \in N$

$\therefore$ Identity element exist, and 1 is the identity element.

Hence, N is a monoid under multiplication.

Example 8

Let $(Z,*)$ be an algebraic structure, where Z is the set of integers and the operation $*$ is defined by $n * m =$ maximum of (n, m). Show that $(Z,*)$ is a semigroups. Is $(Z,*)$ a monoid? Justify your answer.

Solution

Let a, b and c are any three integers.

Closure : Now, $a * b =$ maximum of $(a, b) \in Z$ for all a, b $\in Z$

Associativity: $(a * b) * c =$ maximum of $(a, b, c) = a * (b * c)$

$\therefore$ $(Z,*)$ is a semi groups.

Identity: There is no integer x such that

$a * x =$ maximum of $(a, x) = a$ for all $a \in Z$

$\therefore$ Identity element does not exist.

Hence $(Z,*)$ is not a monoid.

Exercise 1

1. Prove that $(N,\cdot)$ forms a semigroup.
2. Explain why $(N,-)$ is not a semigroup.
3. Is $(Z,+)$ a semigroup? Justify your answer.
4. Determine if $(Z,\bullet)$ forms a semigroup.
5. Show that $(Z_{\geq 0},\bullet)$ is a semigroup.
6. Let R be the set of all real numbers and $*$ is a binary operation defined by $a * b = a + b + a\,b$. Show that $(R,*)$ is a monoid.
7. Let A = Set of all rational numbers $'x'$ such that $0 < x \leq 1$.
   Is $(A,\bullet)$ a monoid and a group?

4.4 Groups

- The algebraic structure in a group is one of the simplest and most fundamental structures in abstract algebra. A group consists of a set combined with a binary operation that satisfies a specific set of properties or operations.

- An algebraic system $(S,*)$ is said to be a group if satisfies the following properties:
  1) $*$ is a **closed** operation.
  2) $*$ is an **associative** operation.
  3) There is an **identity** in S.
  4) Every element in S has **inverse** in S.

- Abelian group (Commutative group): A group $(G,*)$ is said to be abelian (or commutative) if $a * b = b * a$ for all $a, b \in G$.

Example 9

Show that $(Z, +)$ is an abelian group under addition.

Solution

Let Z = set of all integers.
Let a, b, c are any three elements of Z.

Closure: We know that, Sum of two integers is again an integer.
e.g. $a + b \in Z$ for all $a, b \in Z$

Associativity: We know that addition of integers is associative.
e.g. $(a + b) + c = a + (b + c)$ for all $a, b, c \in Z$.

Identity: We have $0 \in Z$ and $a + 0 = a$ for all $a \in Z$.
∴ Identity element exists and '0' is the identity element.

Inverse: To each $a \in Z$, we have $-a \in Z$ such that $a + (-a) = 0$
∴ Each element in Z has an inverse.

Commutativity: We know that addition of integers is commutative.
e.g. $a + b = b + a$ for all a, b E Z.
Hence, $(Z, +)$ is an abelian group.

Let's Solve Together

Exercise 2

1. Verify if $(Z, +)$ forms a group.
2. Show that set of all real numbers `R' is not a group under multiplication.
3. Prove that $(Z, -)$ is not a group.
4. Is $(N, +)$ a group?
5. Prove that $(R, \bullet)$ is an abelian group with respect to multiplication.
6. Show that $(Q, \bullet)$ is a group with respect to multiplication.

Exercise 2

7. Show that the set of all positive rational numbers forms an abelian group under the composition $*$ defined by $a * b = (ab)/2$

8. If M is set of all non singular matrices of order $'n\ x\ n'$, then show that M is a group under matrix multiplication. Is $(M,*)$ an abelian group? Justify your answer.

9. If $E = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$, prove that the algebraic structure $(E, +)$ is an abelian group.

10. Let C = Set of all non zero complex numbers. Prove that $(C,\bullet)$ is an abelian group.

TOPIC 5:
GRAPHS

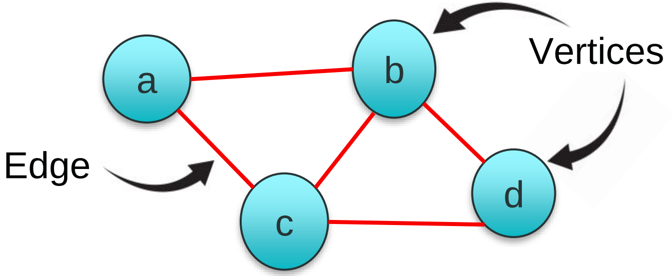
Contents of this topic

5.1
Introduction to graphs

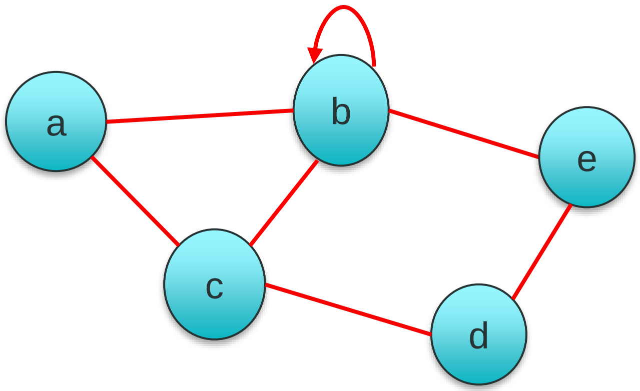
A graph is a pair (V, E), where $\boldsymbol{v}$ is a set of objects called **vertices** and that $\mathbf{E}$ is a set of two element subsets of v known as **edges**.

Important terminology and notation

- ✓ The terms and notations below will show up repeatedly.
- ✓ We will usually denote **vertices** with single letters like a or b.
- ✓ We will denote **edges** with pairs of letters, like ac OR ordered pair $(\boldsymbol{a}, \boldsymbol{c})$ to denote the edge between vertices a and b.
- ✓ We will denote **graphs** with capital letters, like G or H.



For Example, given the graph, G below

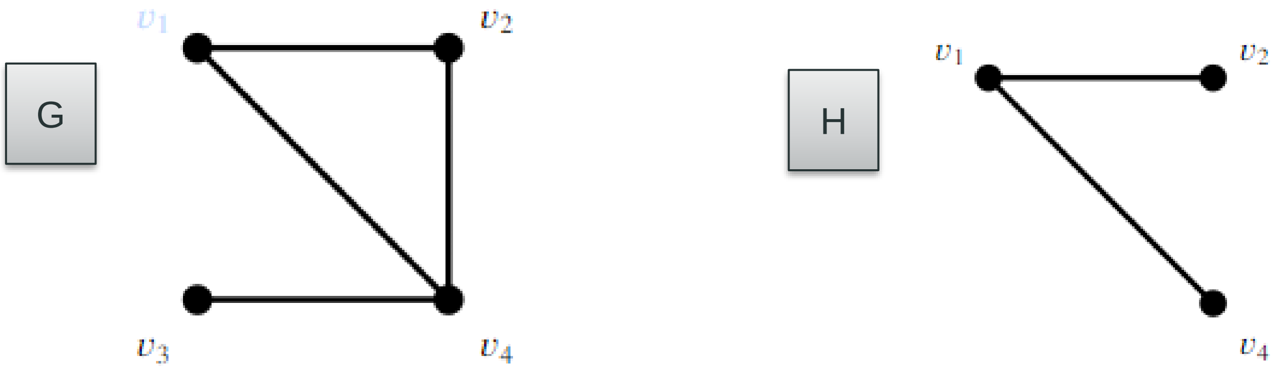


- It is a pair $\mathrm{G} = \{V.E\}$ where V = set of vertices and E = set of edges. So,
- $Vertex, V = \{a, b, c, d, e\}$
- $Edges, E = \{(a, b), (b, a), (a, c), (c, a), (b, e), (e, b), (e, d), (d, e), (c, d), (d, c), (c, b), (b, c)(b, b)\}$

5.1.1 Subgraph

Graph $\mathrm{H} = (\boldsymbol{W}, \boldsymbol{F})$ is a subgraph of graph $\mathrm{G} = (\boldsymbol{V}, \boldsymbol{E})$ if $W \subseteq V$ and $\mathrm{F} \subseteq E$.

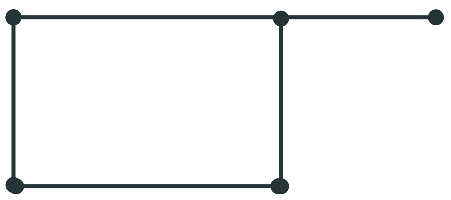
(Since H is a graph, the edges in F have their endpoints in $\boldsymbol{W}$).



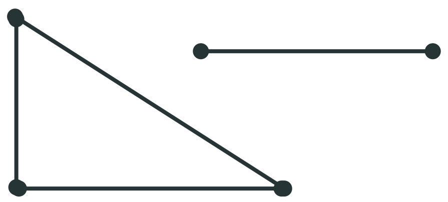
5.1.2 Connected Graphs

A graph is called connected if there is a path from any vertex to any other vertex in the graph. Otherwise the graph is disconnected.

Connected



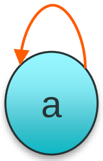
Disconnected – 2 components



5.2
Relations and digraph

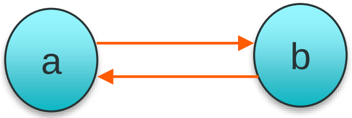
Reflexsive

There is a loop for every vertices



Symmetric

There is an edge from vertex a to b there there is another edge connecting from b to a

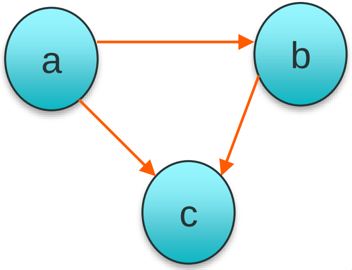


Antisymmetric

There is one edge connected to two vertices a and b

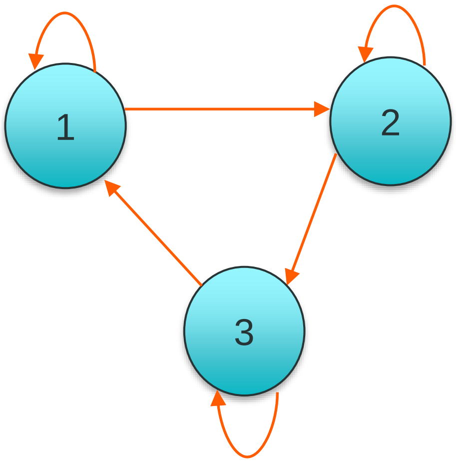
Transitive

There is one edge connected to vertices a and b, and another edge connected to b and c then there is an edge connected to a and c



Example 1

Find relation, R for the following digraph. State whether the following digraph is an equivalence relation or not



✓ R is reflexive since there is a loop for every vertices

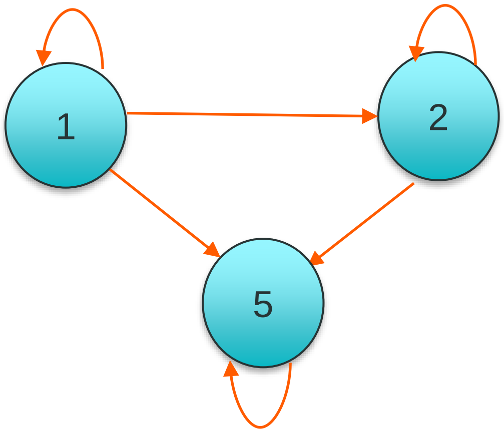
X R is not symmetric since there is only one edge from vertex 1 to 2

X R is not transitive since $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$

∴ the digraph is not equivalence relation.

Example 2

Find relation, R for the following digraph. State whether the following digraph is an equivalence relation or not.



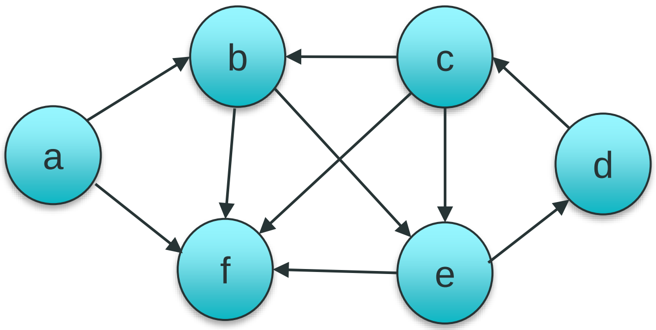
5.3

Euler paths and circuits

Path is a sequence of edges

Item	Definition
Size	Number of vertex in a graph
length	Number of edges in a graph
In Degree	Number of edge which come in into one vertex
Out degree	Number of edge which come out from one vertex
Degree	Number of edge which connected from a vertex
Simple Path	From V and W is a path from V to W with no repeated vertices
Cycle (or circuit)	A path of nonzero length from V to V with no repeated edges
Simple cycle	Is a cycle from V to V in which, except for the beginning and ending vertices that are both equal to V, there is no repeated vertices.

Example 3



Size = 6

Length = 10

	a	b	c	d	e	f
In degree	0	2	1	1	2	4
Out degree	2	2	3	1	2	0
Degree	2	4	4	2	4	4

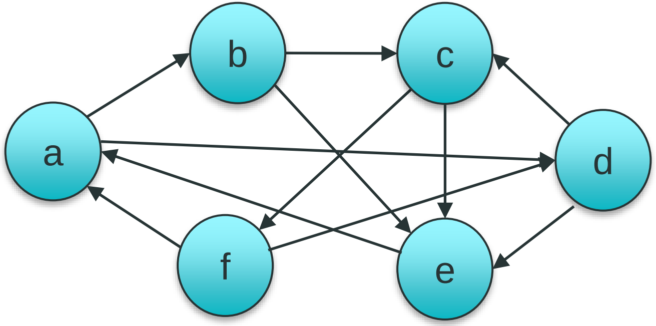
$P_1: b \rightarrow e \rightarrow d \rightarrow c \rightarrow e \rightarrow f$

$P_2: a \rightarrow b \rightarrow e \rightarrow d \rightarrow c \rightarrow f$

$P_3: a \rightarrow b \rightarrow e \rightarrow d \rightarrow c \rightarrow b \rightarrow e \rightarrow f$

Which one is a simple path?

Example 4



$C_1: a \to b \to e \to a$
$C_2: e \to a \to b \to c \to f \to a \to d \to e$
$C_3: a \to b \to c \to f \to d \to e \to a$

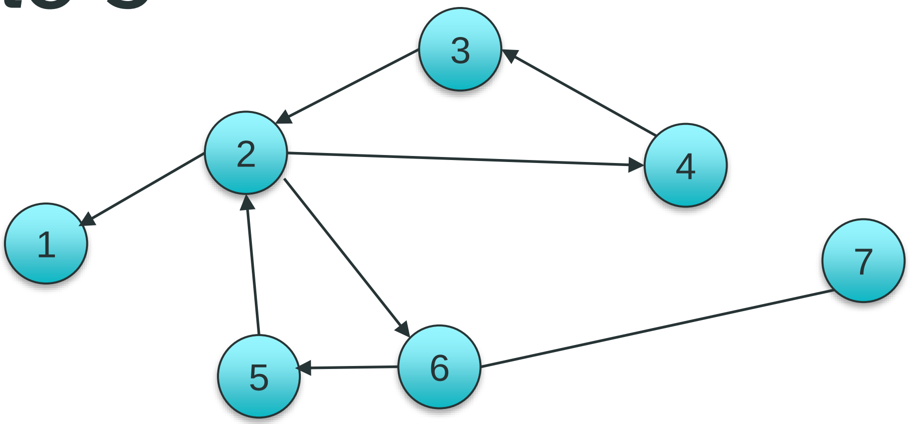
Which one is a simple cycle?

Size =

Length =

	a	b	c	d	e	f
In degree						
Out degree						
Degree						

Example 5



Path	Simple path	Cycle	Simple Cycle
(6,5,2,4,3,2,1)			
(6,5,2,6)			
(6,5,2,4)			
(2,6,5,2,4,3,2)			
(5,6,2,5)			

Euler Circuit and Euler path

Euler Circuit

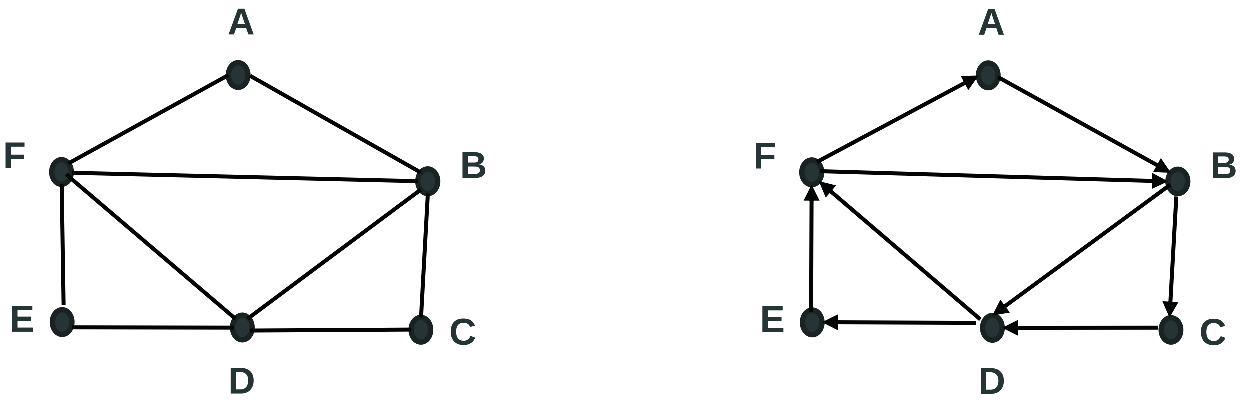
1. An **Euler circuit** is a circuit that uses **every edge** in a graph with no repeats. Being a circuit, it must start and end at the same vertex.
   For example, 0, 2, 1, 0, 3, 4, 0 is an Euler circuit
2. Theorem: A graph will contain an Euler circuit if all vertices have **even degree.**

Euler Path

1. An **Euler path** is a path that uses **every edge** in a graph with no repeats. Being a path, it does not have to return to the starting vertex.
   For example, 0, 2, 1, 0, 3, 4 is an Euler path.
2. Theorem: A graph will contain an Euler path if it contains **at most two vertices of odd degree.**

Example 6

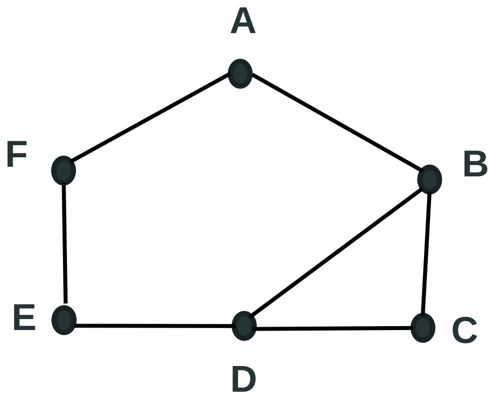
The graph below has several possible Euler circuits. Here's a couple, starting and ending at vertex A: ABCDEFBDFA and AFBDFEDCBA. The second is shown in arrows.



Example 7

In the graph below, vertices A, C, E, and F have degree of 2, since there are 2 edges leading into each vertex and vertices B and D have 3 degree since there are 3 edges leading into each vertex.

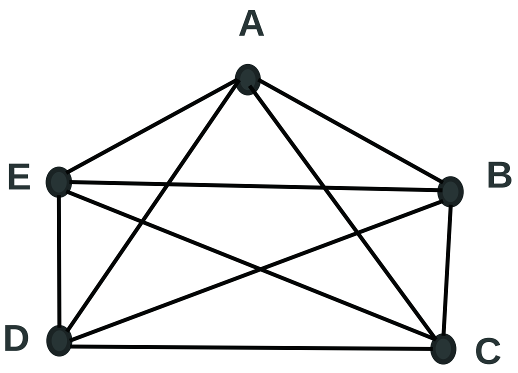
So, This graph contains two vertices with odd degree (B and D) and four vertices with even degree (A, C, E, and F), so Euler's theorems tell us this graph has an Euler path, but not an Euler circuit.



Example of Euler Path: $B \to C \to D \to E \to F \to A \to B \to D$

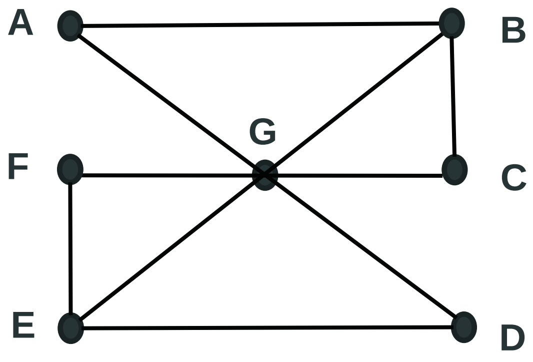
Example 8

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



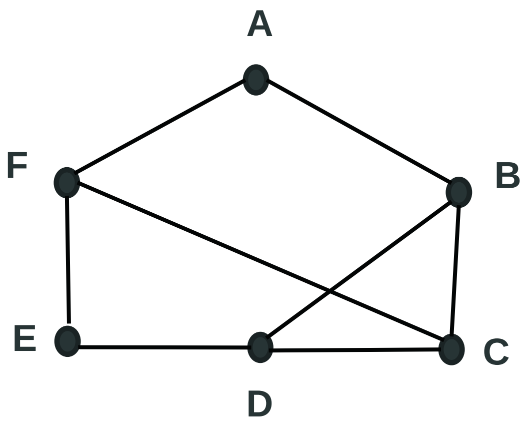
Example 9

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



Example 10

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



5.4

Hamiltonian paths and circuits

Hamiltonian Circuit and Path

Hamiltonian Circuit

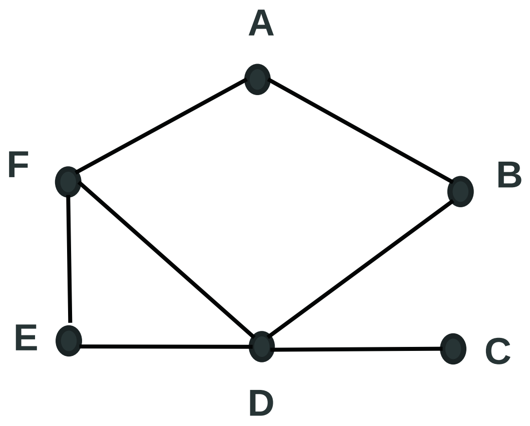
1. A **Hamiltonian circuit** is a circuit that visits **every vertex** once with no repeats. Being a circuit, it must start and end at the same vertex.
2. For example, 0, 2, 1, 3, 4, 0 is an Hamiltonian circuit

Hamiltonian Path

1. A **Hamiltonian path** also visits **every vertex** once with no repeats, but does not have to start and end at the same vertex.
2. For example, 0, 2, 1, 3, 4 is an Hamiltonian path.

Example 11

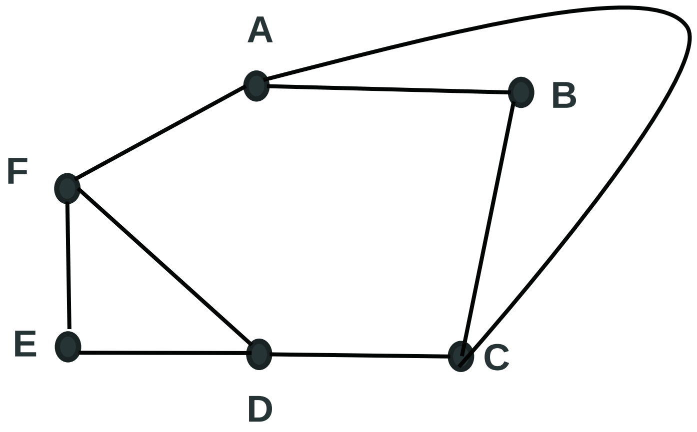
Does a Hamiltonian path or circuit exist on the graph below?



- We can see that once we travel to vertex C there is no way to leave without returning to D, so there is no possibility of a Hamiltonian circuit.
- If we start at vertex C we can find several Hamiltonian paths, such as CDEFAB and CDBAFE

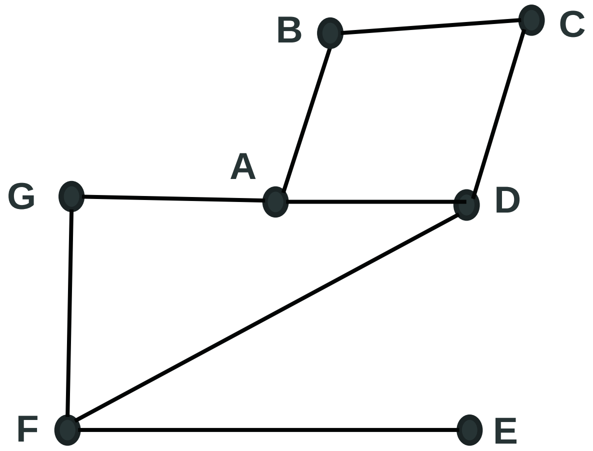
Example 12

Determine whether the given graph has an Hamilton circuit and Hamiltonian path. Construct such a circuit and path if one exists.



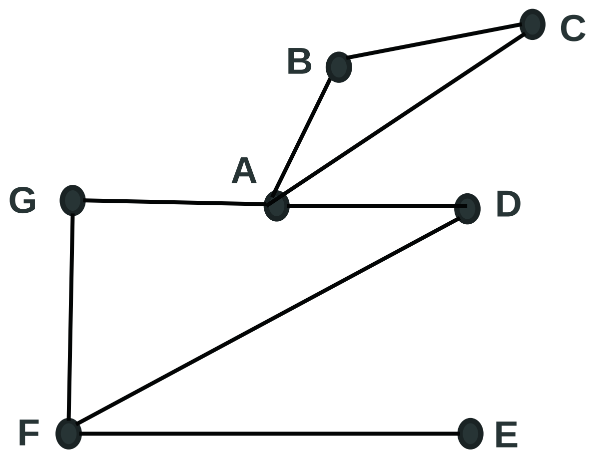
Example 13

Determine whether the given graph has an Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



Example 14

Determine whether the given graph has an Hamilton circuit. Construct such a circuit when one exists. If no Hamilton circuit exists, determine whether the graph has an Hamilton path and construct such a path if one exists.



Example 15

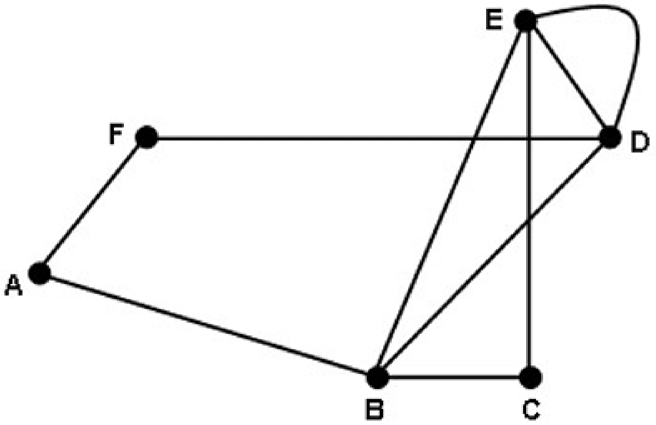
Determine the:

a. Size and length of the graph.

b. Euler path or circuit if it exists.

c. Hamiltonian circuit. If there is none, give your reason(s).

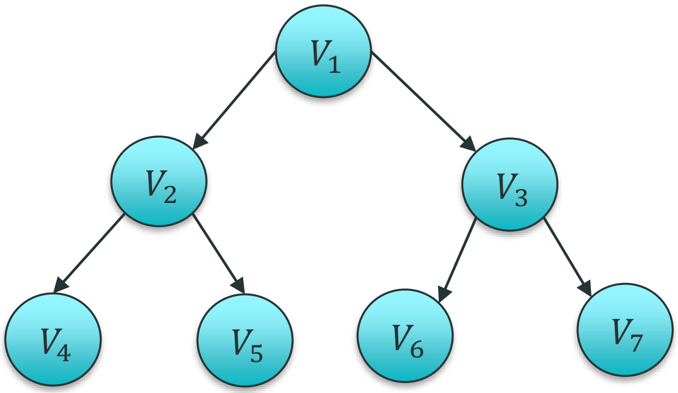
d. Hamiltonian path, If there is none, give your reason(s).



Category	Path	Circuit
Euler	1. MUST Containing every **EDGE** exactly once	1. MUST Containing every **EDGE** exactly once
	2. Have exactly only **TWO** vertices of **ODD degree**	2. Every vertices must only have **EVEN degree**
	3. Must start with **ODD degree vertex** and ending with **another ODD degree vertex**	3. Start with any vertex and ending with the same vertex
	4. Its ok if the circuit have repeating **VERTEX**	4. Its ok if the circuit have repeating **VERTEX**
Hamiltonian	1. Include every **VERTEX** exactly **ONCE**	1. Include every **VERTEX exactly ONCE** except for starting and ending
	2. Starting and ending vertices are not same.	2. Starting and ending vertices are same.

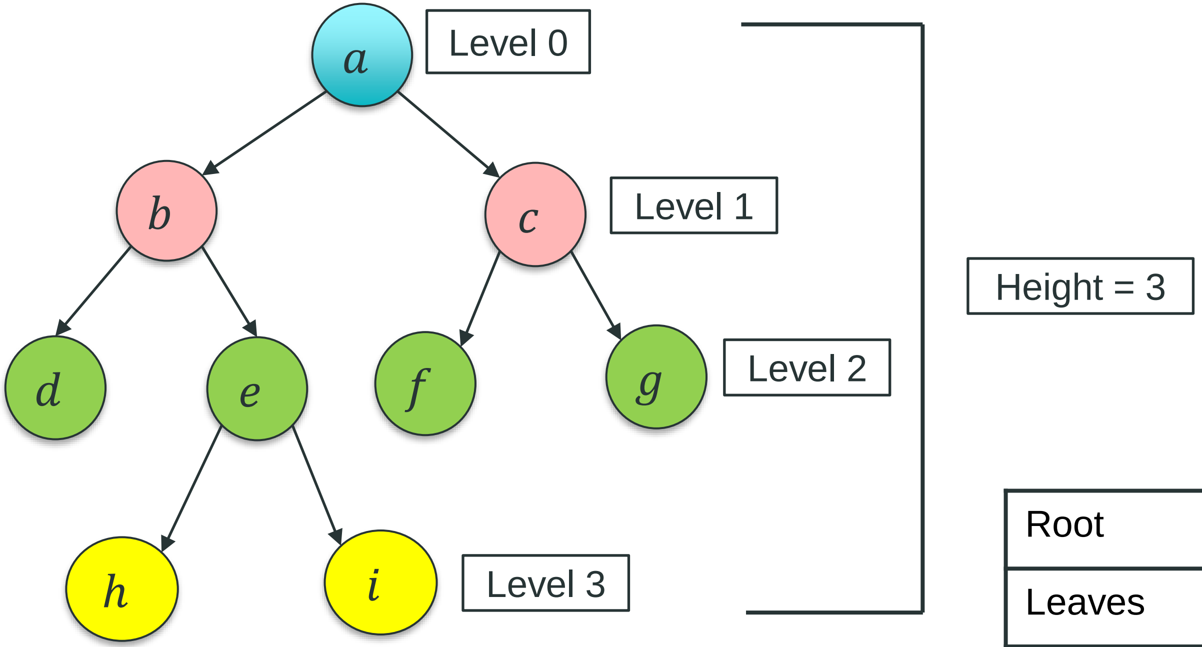
5.5 Trees

A connected graph without cycle.



5.5.1 Rooted Tree

1. A rooted tree is a tree with a special vertex labelled as the "root" of the tree.
2. The root serves as a point of reference for other vertices in the tree. In diagrams, we usually keep the root at the top and list other vertices below it.
3. The **level of a vertex** of a rooted tree is the number of edges that separate the vertex from the root. The level of the root is zero.
4. The **height** of a node is the number of edges from the node to the deepest leaf.
5. A **leaf** is a vertex without any child (no offspring)
6. A **siblings** when two nodes connected to the same node.
7. A **descendant node** is any node in the path from that node to the leaf



Root	a
Leaves	d, h, i, f, g
Sibling of node g	f
Descendant of node b	d, e, h, i
Ascendants of node e	b,a

Example 16

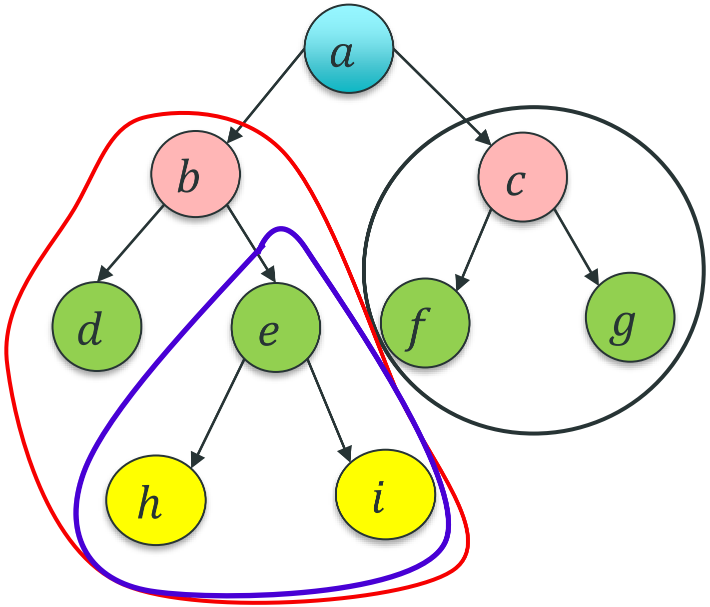
Let $A = \{a, b, c, d, e, f, g, h, i, j\}$

$T = \{(b, c), (b, a), (d, e), (d, f), (e, h), (f, g), (d, b), (g, i), (g, j)\}$.

Show that T is a rooted tree and identify the root

5.5.2 Subtree

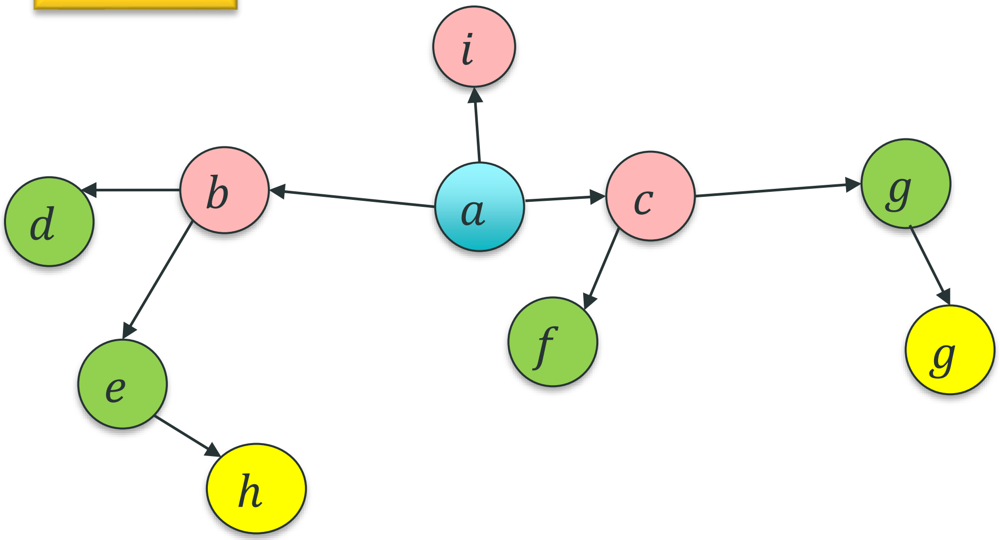
A **subtree** is just a connected subgraph.



5.5.3 Ordered tree

Suppose that the offspring of each vertex of the tree are linearly ordered. Thus, if a vertex v has four offspring, assume that they are ordered, so refer them as the first, second, third or fourth offspring of v. Arrange the offspring from left to right.

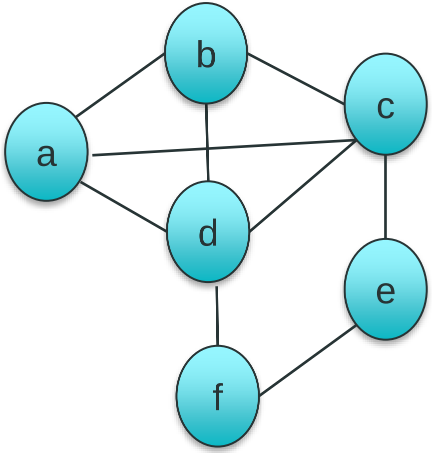
Example Given T as below, if e is a root of T, then draw an ordered tree

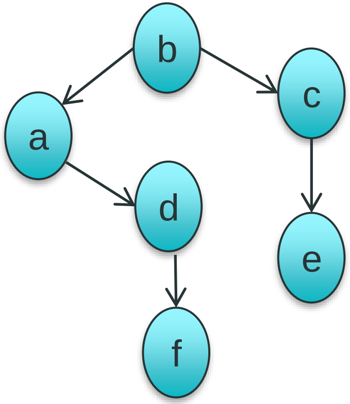


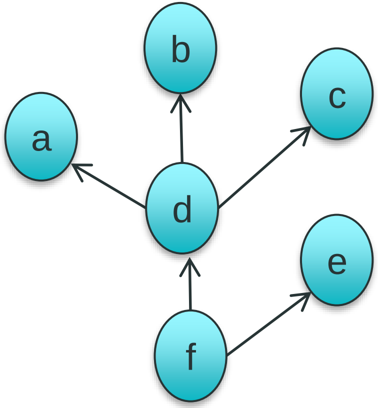
5.5.4 Spanning tree

The tree T is called a spanning tree if T is a subgraph of G having the same vertices as G with minimum edges..

Example Find two directed spanning tree from the graph G

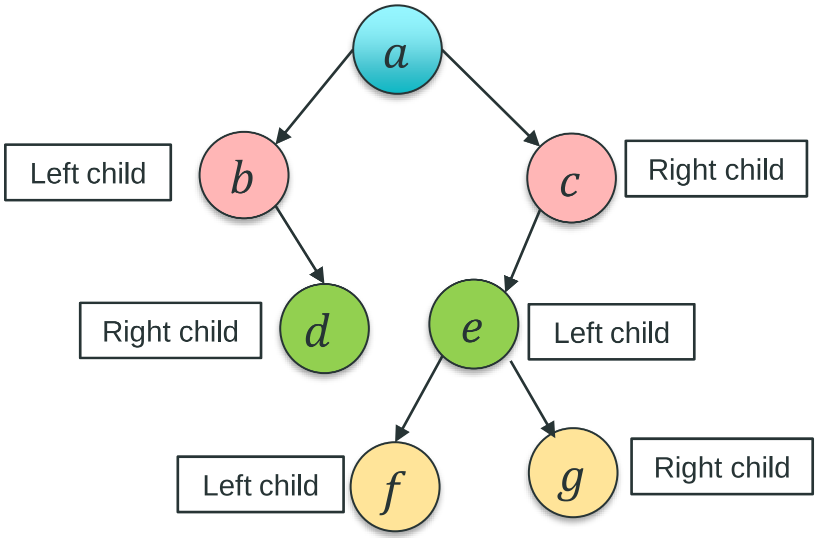






5.5.5 Binary Tree

A rooted tree is a binary tree in case each vertices has at most two children. A left child, a right child, both a left and a right child or no children at all.

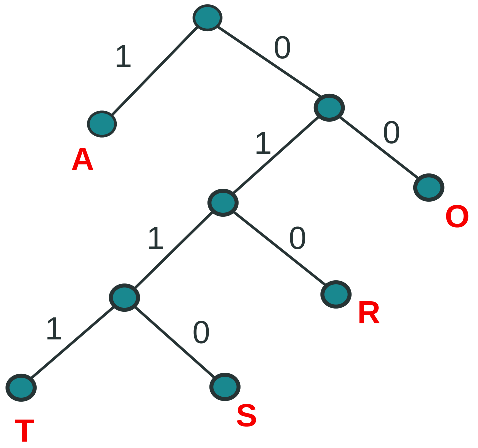


5.5.6 Huffman Tree

- Represent characters by variables-length bit strings. A Huffman code is most easily defined by a rooted tree.
- To decode a bit sting, begin at the root and move down the tree until character is encountered. The bit ‘0’ and ‘1’ tells whether to move right or left

Example Decode each bit string using the Huffman code given below.

a. 01010111
b. 0110000100111

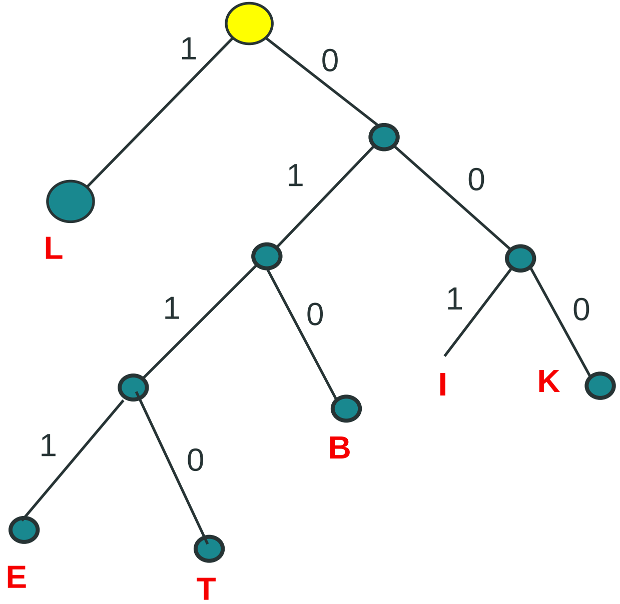


Solution

a. 01010111= RAT
b. 0110000100111 = SORT

Example 17

Decode each bit string using the Huffman code given below



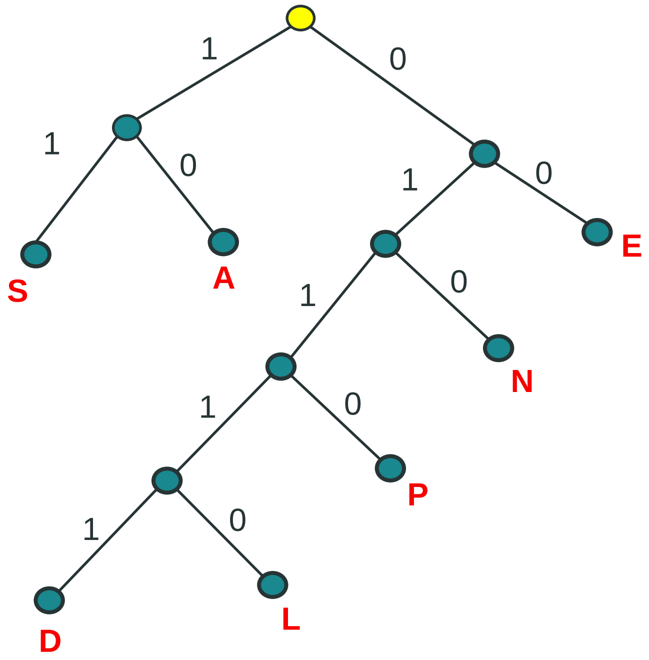
a. 10010000111

b. 00000101100111

c. Find the bit string for code ‘BELT’

Example 18

Decode each bit string using the Huffman code given below



a. 011000010

b. 01110100110

c. 01111001001110

d. 1110011101001111

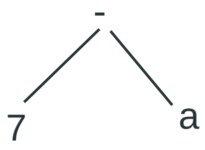
e. Find the bit string for code ‘PENS’

5.5.7 Algebraic Tree

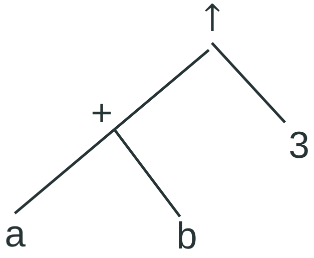
Algebraic expression trees represent expressions that contain numbers, variables, and binary operators

Example Find the binary rooted tree for the following algebraic expression.

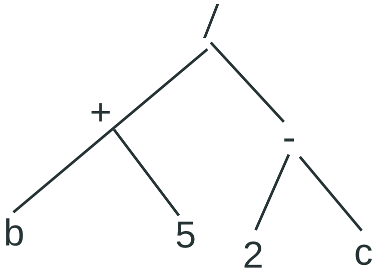
a. $(7 - a)$



$b.$ $(a + b)^3$



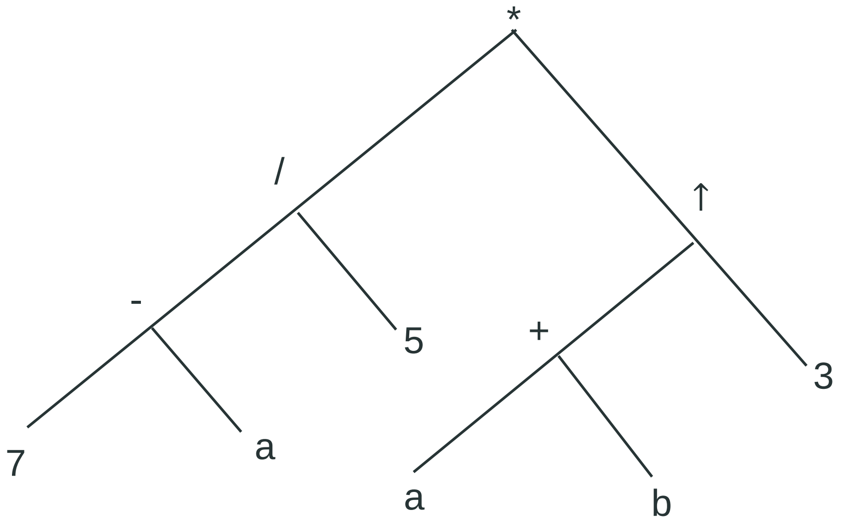
$c.$ $\dfrac{b + 5}{2 - c}$



Example 19

Find the binary rooted tree for the following algebraic expression.

$$\big((7-a)/5\big) * \big((a+b)\uparrow 3\big)$$



Example 20

Find the binary rooted tree for the following algebraic expression.

$$(3 - 2x) + \big((x - 2) - (3 + x)\big)$$

Example 21

Find the binary rooted tree for the following algebraic expression.

$$\frac{(5 - x)}{((6 + 9y) - 3x)}$$

Exercise

1. Find the binary rooted tree for the following algebraic expression.

a. $((5x - 6y) + 4)\left(\dfrac{2y + 3x}{8}\right)$

b. $\dfrac{(3 - (2a + 3b))}{7(4 - b)}$

c. $\dfrac{(xy - 7)}{3(6 - (y + 2))}$

d. $(x^2 + y^2)\dfrac{(2x + 4y^2)}{2}$

e. $\dfrac{(x + 3)^2(x - 2)}{(6x - 5) + xy}$

2. Consider the algebraic expression.

$$\frac{5-\left(3m+\frac{6}{t}\right)}{9(4-n)}$$

a. Draw a rooted tree to represent the given algebraic expression.

b. Determine the root of the tree.

c. List all nodes at level 2

d. List all the leaves.

e. List all the siblings of node ‘m’.

f. List all descendants of the node ‘+’.

g. State the height of the rooted tree.

Thank You!
&
Good Luck

REFERENCES

Fairfield, J. P. (2020). Discrete Mathematics for Computer Science: An Example-Based Introduction (1st ed.). CRC Press LLC.

Hunter, D. J. (2017). Essentials of Discrete Mathematics (3rd ed.). Jones & Bartlett Learning.

Johnsonbaugh, R. (2018). Discrete Mathematics (8th ed.). Pearson Prentice Hall.

Lipschutz, S., & Lipson, M. (2007). Discrete mathematics (3rd ed.). McGraw Hill Education.

Mohammad, S. H., & Roslan, N. S. M. (2021). Discrete mathematics for polytechnic. Politeknik Muadzam Shah.

Rosen, K. H. (2019). Discrete mathematics and its applications (8th ed.). McGraw Hill Education.

Selamat, A., & Selamat, M. H. (2012). Discrete Mathematics for Computer Scientists (1st ed.). Universiti Teknologi Malaysia.

Dummit, D. S., & Foote, R. M. (2004). Abstract algebra (3rd ed.). Wiley.

Artin, M. (2011). Algebra (2nd ed.). Prentice Hall.

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"The world is continuous, but the mind is discrete"
- David Mumford